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APPROXIMATE IDENTITIES AND PARACOMPACTNESS

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Dedicated to Heron S. Collins, our teacher and friend

ABSTRACT. Let X denote a locally compact Hausdorff space and $C_b(X)$ the algebra of continuous complex-valued functions on X. The main result of this paper is that X is paracompact if and only if $C_0(X)$, the subalgebra of $C_b(X)$ consisting of functions which vanish at infinity, has an approximate identity which is a relatively compact subset of $C_b(X)$ for the weak topology of the pairing of $C_b(X)$ with its strict topology dual.

Throughout this paper X denotes a locally compact Hausdorff space, $C_b(X)$ denotes the algebra of continuous complex-valued functions on X, equipped with the strict topology β introduced by Buck [**1**] and studied by numerous others (for example, see [**5**, **11**], and the surveys [**2**, **14**]), and $C_0(X)$ denotes the algebra of continuous complex-valued functions which vanish at infinity on X. Buck proved that the β-dual of $C_b(X)$ is $M_t(X)$, the space of bounded tight (inner regular by compact sets) Borel measures on X. By the weak topology on $C_b(X)$, we mean the topology $\sigma(C_b(X), M_t(X))$.

An approximate identity for $C_0(X)$ is a uniformly bounded net $(f_\alpha)_{\alpha\in A}$ in $C_0(X)$ such that $\|hf_\alpha - h\| \to 0$ for each $h \in C_0(X)$. An approximate identity is said to be *well-behaved* (a WBAI) if the following conditions hold: (1) $0 \le f_\alpha \le 1$ for all $\alpha \in A$; (2) $\alpha_1 < \alpha_2 \Rightarrow f_{\alpha_1} f_{\alpha_2} = f_{\alpha_1}$; and (3) if $\alpha \in A$ and (α_n) is a strictly increasing sequence in A, then there exists a positive integer N such that $f_\alpha f_{\alpha_n} = f_\alpha f_{\alpha_N}$ for $n \ge N$. Taylor [**12**] introduced the notion of WBAI and showed that if X is paracompact, then $C_0(X)$ has a WBAI. We shall use two other abbreviations: WCAI, to stand for an approximate identity for $C_0(X)$ that is relatively weakly compact in $C_b(X)$, and TBAI, to denote a β-totally bounded (= equicontinuous) approximate identity for $C_0(X)$.

Collins and Fontenot [**4**] studied various classes of approximate identities for $C_0(X)$, relationships between these classes, and implications for the topological properties of X. Building on earlier work of Collins and Dorroh [**3**], they showed that $C_0(X)$ has a TBAI if and only if X is paracompact. Subsequently, Wheeler [**13**] showed that paracompactness is also equivalent to existence of a WBAI for $C_0(X)$. His argument uses Stone-Čech compactifications and a set-theoretic lemma due to A. Hajnal. Collins and Fontenot also considered WCAI's: they gave an example, the space of countable ordinals, of a space X such that $C_0(X)$ has no WCAI and left open the problem of characterizing spaces X such that $C_0(X)$ has a WCAI.

In this paper we do three things. First, and principally, we show that $C_0(X)$ has a WCAI if and only if X is paracompact, and we observe that our argument also

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applies in the WBAI case to give a new and simpler proof of Wheeler's result [**13**, Theorem 2.1]. Second, by a direct argument that does not appeal to paracompactness of X, we prove that if $C_0(X)$ has a WCAI, then it has a WBAI and note that this result gives a second proof of our main theorem. Since a TBAI is a WCAI, we also get a new proof that $C_0(X)$ has a TBAI if and only if X is paracompact. Third, we give an example to show that a WBAI or WCAI need not contain a subnet which is a TBAI. Thus we show that a plausible approach to both the WBAI and WCAI characterization problems will not work.

We begin by mentioning two (essentially) known results about relatively weakly compact sets in $C_b(X)$. The first is this: A uniformly bounded subset F of $C_b(X)$ is relatively weakly compact if and only if it is relatively pointwise compact if and only if it is quasi-equicontinuous [**7**, p. 269]. The second result is the content of the following lemma.

LEMMA 1. *A uniformly bounded subset F of $C_0(X)$ is relatively weakly compact in $C_b(X)$ if and only if every sequence in F has a subsequence converging pointwise to a function in $C_b(X)$.*

PROOF. Suppose that F satisfies the condition on sequences. The Lebesgue Dominated Convergence Theorem then implies that every infinite subset of F has a weak cluster point in $C_b(X)$. Since β is complete, F is relatively weakly compact [**8**, Proposition 2, p. 177].

Suppose that F is relatively weakly compact. Then F is certainly relatively countably compact in the topology of pointwise convergence on $C_b(X)$. Let (f_n) be a sequence in F, and let $S = \bigcup_n \{x \in X\colon f_n(x) \neq 0\}$. Note that S is σ-compact. Let $F \in C_b(X)$ be a pointwise cluster point of the sequence (f_n). By [**9**, Theorem 2.1], there is a subsequence (f_{n_k}) such that $f_{n_k}(x) \to f(x)$ for $x \in S$. But $f_{n_k}(x) = f(x) = 0$ for $x \in X - S$. Thus $f_{n_k} \to f$ pointwise on X. This concludes the proof.

The proof of Theorem 3, our main result, uses a set-theoretical lemma due to Hajnal (see [**13**, Lemma 2.6]). We give here a much simpler proof due to Ruzsa [**10**].

LEMMA 2. *Let A be an infinite set, directed by $<$. Let $(K_\alpha)_{\alpha\in A}$ be a cover of a set X with the following properties:* (a) *if $\alpha_1 < \alpha_2$, then $K_{\alpha_1} \subset K_{\alpha_2}$;* (b) *if (α_n) is a strictly increasing sequence in A and $\alpha \in A$, then there is a positive integer N such that $K_\alpha \cap K_{\alpha_n} = K_\alpha \cap K_{\alpha_N}$ for $n \geq N$. Then there is a set B and a family $(L_\beta)_{\beta\in B}$ of pairwise disjoint subsets of X which covers X and has the following properties:* (1) *for each $\beta \in B$, $\exists \alpha \in A$, such that $L_\beta \subset K_\alpha$;* (2) *for each $\alpha \in A$, there is a finite subset F of B such that $K_\alpha \subset \bigcup_{\beta\in F} L_\beta$.*

PROOF. Let $\ll$ be an arbitrary well-ordering of A. Let B be the set of finite, nonempty subsets of A. Define a well-ordering, also denoted $\ll$, of B as follows. Let $\beta = \{\alpha_1, \ldots, \alpha_m\}$ and $\beta' = \{\alpha'_1, \ldots, \alpha'_n\}$, with $\alpha_m \ll \alpha_{m-1} \ll \cdots \ll \alpha_1$ and $\alpha'_n \ll \alpha'_{n-1} \ll \cdots \ll \alpha'_1$, be two different elements of B. If $\alpha_i = \alpha'_i$ for every $i \leq \min(m,n)$, let $\beta \ll \beta'$ if and only if $m < n$. If not, let j be the first index for which $\alpha_j \neq \alpha'_j$, and let $\beta \ll \beta'$ if and only if $\alpha_j \ll \alpha'_j$. Note that if $\beta \subset \beta'$, then $\beta \ll \beta'$.

Define, by induction on the cardinality of $\beta \in B$, an auxiliary function $g\colon B \to A$ in such a way that $g(\{\alpha\}) = \alpha$ and if $\beta \subset \beta'$, then $g(\beta) < g(\beta')$. Now let

$$L_\beta = K_{g(\beta)} - \bigcup_{\beta' \ll \beta} K_{g(\beta')}.$$

The family (L_β) consists of disjoint sets, has the same union as (K_α), and has the property that $L_\beta \subset K_{g(\beta)}$. All we need to prove is (2).

We prove by contradiction that each $K_{g(\beta)}$ is contained in a finite union of sets in the family (L_β); this is sufficient for (2) since $K_\alpha = K_{g(\{\alpha\})}$. Suppose not, and let $\beta = \{\alpha_1, \ldots, \alpha_n\}$, with $\alpha_n \ll \cdots \ll \alpha_1$, be the *first* element of B for which our proposition does not hold. Since $K_{g(\beta)} - L_\beta \subset \bigcup_{\beta' \ll \beta} L_{\beta'}$, there must be an infinite sequence $\beta_1 \ll \beta_2 \ll \cdots \ll \beta$ such that $K_{g(\beta)} \cap L_{\beta_i} \neq \varnothing$ for all i. Divide these β_i's into n classes, the first consisting of those with all elements $\ll \alpha_1$ and the j-th, for $2 \leq j \leq n$, consisting of those β_i's which contaian $\alpha_1, \ldots, \alpha_{j-1}$ but not α_j. One of these classes, say the kth, must be infinite. Denote the β_i's of this class by $\delta_1 \ll \delta_2 \ll \cdots$.

Now let $\gamma_i = \delta_1 \cup \cdots \cup \delta_i$. Then $\gamma_i \ll \beta$ since γ_i contains $\alpha_1, \ldots, \alpha_{k-1}$ and all its other elements are $\ll \alpha_k$. Since $\gamma_1 \subset \gamma_2 \subset \cdots$, $g(\gamma_1) < g(\gamma_2) < \cdots$. Since $(K_\alpha)_{\alpha \in A}$ satisfies condition (b), there exists a positive integer N such that for $n \geq N$

$$K_{g(\beta)} \cap K_{g(\gamma_n)} = K_{g(\beta)} \cap K_{g(\gamma_N)}.$$

Thus, for $n \geq N$

$$\begin{aligned}(L_{\delta_n} \cap K_{g(\gamma_N)}) &\supset (L_{\delta_n} \cap K_{g(\gamma_N)} \cap K_{g(\beta)}) = (L_{\delta_n} \cap K_{g(\gamma_n)} \cap K_{g(\beta)}) \\ &\supset (L_{\delta_n} \cap K_{g(\delta_n)} \cap K_{g(\beta)}) = L_{\delta_n} \cap K_{g(\beta)} \neq \varnothing.\end{aligned}$$

Since the sets in the family $(L_{\beta'})_{\beta' \in B}$ are pairwise disjoint, this implies that $K_{g(\gamma_N)}$ is not contained in a finite union of sets in the family. This is a contradiction since $\gamma_N \ll \beta$.

THEOREM 3. *Let X be a locally compact Hausdorff space. Then $C_0(X)$ has a WCAI if and only if X is paracompact.*

PROOF. If X is paracompact, the construction in [**3**, Theorem 4.2] yields a WCAI, since β-totally bounded sets are relatively β-compact.

Now suppose that $(f_\alpha)_{\alpha \in A}$ is a WCAI for $C_0(X)$. For each $\alpha \in A$, let $K_\alpha = \{x\colon f_\alpha(x) \geq \frac{1}{2}\}$. The family (K_α) is a cover of X consisting of compact sets. Introduce a (new) partial order on A as follows: $\alpha_1 < \alpha_2$ if and only if $f_{\alpha_2} > \frac{3}{4}$ on K_{α_1}. It is evident that A is a directed set and that if $\alpha_1 < \alpha_2$, then $K_{\alpha_1} \subset K^0_{\alpha_2} \subset K_{\alpha_2}$. Suppose that (α_n) is a strictly increasing sequence in A with respect to the partial order just defined. Let $\Sigma = \bigcup_n K_{\alpha_n}$, and let f be a weak and, hence, pointwise cluster point of (f_{α_n}) in $C_b(X)$. Since $f \geq \frac{3}{4}$ on Σ and $f \leq \frac{1}{2}$ on $X - \Sigma$, Σ is clopen.

Let K be any compact subset of X, e.g., K_α for some α. Then $K \cap \Sigma$ is compact, and there exists a positive integer N such that $K \cap \Sigma = K \cap K_{\alpha_N}$ since $\Sigma = \bigcup_n K^0_{\alpha_n}$. Therefore $K \cap K_{\alpha_n} = K \cap K_{\alpha_N}$ for $n \geq N$.

We now choose sets $(L_\beta)_{\beta \in B}$ as in Lemma 2 and observe that this family of sets is a locally finite (called neighborhood-finite in [**6**]) cover of X by pairwise disjoint, relatively compact sets. It follows easily that every open cover of X has a (not necessarily open) locally finite refinement; thus X is paracompact [**6**, Theorem 2.3, p. 163].

REMARK. A slight modification of the preceding argument furnishes a new and simpler proof of Wheeler's WBAI theorem [**13**, Theorem 2.1]. Let $(h_\alpha)_{\alpha\in A}$ be a WBAI and, for each α, let K_α be the (compact) support of the function h_α. Let $<$ denote the given partial order on A. It is easy to see that Lemma 2 applies to the net (K_α). The preceding argument shows that X is paracompact.

The next result does not invoke paracompactness of X, so it yields another proof of Theorem 3 via [**13**, Theorem 2.1].

PROPOSITION 4. *Let X be a locally compact Hausdorff space. If $C_0(X)$ has a WCAI, then it has a WBAI.*

PROOF. Let $(f_\alpha)_{\alpha\in A}$ be a WCAI for $C_0(X)$ and, for each α, let $K_\alpha = \{x\colon f_\alpha(x) \geq \frac{1}{2}\}$, $B_\alpha = \{x\colon f_\alpha(x) > \frac{1}{3}\}$, and $C_\alpha = \{x\colon f_\alpha(x) \geq \frac{1}{3}\}$. Choose, for each α, a continuous function h_α on X such that $0 \leq h_\alpha \leq 1$, $h_\alpha = 1$ on K_α, and $h_\alpha = 0$ on $X - B_\alpha$. Note that the (compact) support of h_α is a subset of C_α. Put a (new) partial ordering $<$ on A as follows: $\alpha_1 < \alpha_2$ if and only if $f_{\alpha_2} > \frac{3}{4}$ on C_{α_1}.

To show that $(h_\alpha)_{\alpha\in A}$ is a WBAI, we need only check part (3) of the definition, since the rest of the definition clearly holds. Let $\alpha \in A$, K be the support of h_α, and (α_n) be a strictly increasing sequence in A. As in the proof of Theorem 3, let $\Sigma = \bigcup_n K_{\alpha_n} = \bigcup_n K^0_{\alpha_n}$ and conclude that, for some positive integer N, $K \cap \Sigma = K \cap K_{\alpha_N}$. Thus if $x \in K \cap \Sigma$, then $h_{\alpha_n}(x) = h_{\alpha_N}(x) = 1$ for $n \geq N$. If $x \in K - \Sigma$, then $h_{\alpha_n}(x) = 0$ for all n since Σ contains the support of h_{α_n} for all n. Thus $h_\alpha h_{\alpha_n} = h_\alpha h_{\alpha_N}$ for $n \geq N$ and, therefore, $(h_\alpha)_{\alpha\in A}$ is a WBAI.

We have seen that X paracompact $\Rightarrow$ existence of a TBAI $\Rightarrow$ existence of a WCAI $\Rightarrow$ existence of a WBAI $\Rightarrow$ X is paracompact. It is natural to inquire if every WBAI or WCAI contains a subnet which is a TBAI or, equivalently, which is equicontinuous. The concluding example shows that this need not be the case.

EXAMPLE 5. Let N denote the set of positive integers, and let X be the topological sum (union of disjoint spaces, each clopen in X) of the product space N^N, given the discrete topology, and sets $A_1, A_2, \ldots$, each of which is a copy of the closed unit interval $[0,1]$ with the usual topology. Observe that X is locally compact Hausdorff and paracompact [**6**, Theorem 7.3, p. 241]. For each $n \in N$ and for each $x \in [0,1]$, let $h_n(x) = nxe^{-nx}$. Note that $h_n\colon [0,1] \to [0, 1/e]$ for each n and that $h_n \to 0$ pointwise. Since $h_n(1/n) = 1/e$, no subsequence of (h_n) is equicontinuous at $x = 0$.

Let us introduce some notation. Let $\pi_n\colon N^N \to N$ be the projection map onto the nth coordinate for each n. For each finite subset $S = \{a_1, \ldots, a_n\}$ of N^N, let $\varphi(S) = \sum\sum \pi_i(a_j)$ where the double summation is over the range $1 \leq i \leq n$, $1 \leq j \leq n$. Let $f_S\colon X \to [0,1]$ be defined as follows: $f_S = 1$ on $S \cup A_1 \cup \cdots \cup A_n$, $f_S = h_{\varphi(S)}$ on A_{n+1}, $f_S(x) = 0$ for other x.

It is clear that each function f_S is continuous and has compact support and that the net (f_S), where the finite subsets S are directed upward by inclusion, is an approximate identity for $C_0(X)$. Using a diagonal process, one can easily see that each sequence from the family (f_S) has a subsequence which converges pointwise to an element of $C_b(X)$. Using Lemma 1, we conclude that (f_S) is a WCAI. It is also easily checked that (f_S) is a WBAI.

Now let $F \subset (f_S)$ be any equicontinuous subset. Because no infinite subset of the family (h_n) is equicontinuous, for each $n \in N$ there exist $k_n \in N$ such that if $S = \{a_1, \ldots, a_n\} \subset N^N$ and $\varphi(S) \geq k_n$, then $f_S \notin F$. Let $x = (k_1, k_2, \ldots)$ and suppose that $f_S(x) \neq 0$ for some S of cardinality n. Then $x \in S$ and $\varphi(S) \geq k_1 + \cdots + k_n \geq k_n$. Therefore $f_S \notin F$. Thus $f_S(x) = 0$ for every $f_S \in F$ and, hence, F is not an approximate identity for $C_0(X)$.

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