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ON AN EIGENVALUE PROBLEM OF AHMAD AND LAZER FOR ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. In connection with a problem posed by S. Ahmad and A. C. Lazer, we show the existence of a class of nonselfadjoint eigenvalue problems related to the equation $y^{(n)} + \lambda p(x)y = 0$ for which the general eigenvalues comparison is not true. We use a comparison principle for the zeros of the corresponding Cauchy problem.

This paper provides a contribution to the understanding of a problem raised by S. Ahmad and A. C. Lazer [**1**] in connection with the comparison of the eigenvalues for some multi-point boundary value problems which are not selfadjoint.

One is given the equation

$$L_n y + \lambda p(x)y = 0, \tag{1}$$

where $p(x)$ is a continuous function of constant sign on an interval I, λ is a parameter, and $L_n y$ is a linear differential disconjugate operator of order n, that is, the only solution of $L_n y = 0$ with n zeros on I (counting multiplicity) is $y \equiv 0$.

Let us consider the eigenvalue problem given by equation (1) and the system of boundary conditions

$$\begin{aligned} L_i y(a) &= 0, \qquad i \in \{i_1, \dots, i_k\}, \\ L_j y(b) &= 0, \qquad j \in \{j_1, \dots, j_{n-k}\}, \end{aligned} \tag{2}$$

where $a, b \in I$, $1 \leq k \leq n-1$, $L_i y$, $i = 0, \dots, n-1$, are the quasi-derivatives of $y(x)$ (see [**7**]), and $\{i_1, \dots, i_k\}$, $\{j_1, \dots, j_{n-k}\}$ are two arbitrary sets of indices from the set $\{0, \dots, n-1\}$.

Problems of this type have been studied extensively (cf. [**2**, **3**, **5**]). In particular, Elias [**5**] has shown that if $(-1)^{n-k}p(x) < 0$, then the eigenvalues of problems (1) and (2) are real and nonnegative and form a divergence sequence $\{\lambda_m\}_{m \in \mathbf{N}}$.

Ahmad and Lazer [**1**] have considered a particular type of boundary condition (2), that is

$$\begin{aligned} y(a) = y'(a) = \cdots = y^{(k-1)}(a) = 0, \\ y(b) = y'(b) = \cdots = y^{(n-k-1)}(b) = 0, \end{aligned} \tag{3}$$

and showed that if we set $p = p_i$, where p_i, $i = 1, 2$, are two continuous functions, considering the corresponding sequence of eigenvalues $(\lambda_{i,m})_{m \in \mathbf{N}}$, $i = 1, 2$, ordered by magnitude, then the condition

$$(-1)^{n-k}p_2(x) \leq (-1)^{n-k}p_1(x) < 0 \tag{4}$$

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implies that

$$\lambda_{1,1}\lambda_{1,2}\cdots\lambda_{1,m} \geq \lambda_{2,1}\lambda_{2,2}\cdots\lambda_{2,m}$$

for every $m \geq 1$. In the same paper they have raised the question of studying when the condition (4) also implies

$$\lambda_{1,m} \geq \lambda_{2,m} \quad \text{for every } m \geq 1;$$

an assertion that is true in the selfadjoint case, that is when the operator L is selfadjoint, n is even, and $k = n/2$.

This paper aims at pointing out a general class of eigenvalue problems (1), (2) for which the eigenvalues' comparison does not follow from condition (4).

In the following we consider the operator $L_n y = y^{(n)}$ and the case for which only one condition is set at one of the end points a or b, that is $k = 1$ or $k = n-1$. Since for $n = 2$ the problem is selfadjoint, in the following we also suppose that $n \geq 3$. According to this assumption, the problem (1), (2) becomes

$$y^{(n)} + \lambda p(x) y = 0, \tag{5}$$

$$\begin{aligned} y^{(i_1)}(a) &= \cdots = y^{(i_k)}(a) = 0, \\ y^{(j_1)}(b) &= \cdots = y^{(j_{n-k})}(b) = 0 \end{aligned} \tag{6}$$

with $k = 1$ and $(-1)^{n-1}p(x) < 0$ or $k = n-1$ and $p(x) > 0$.

We prove the following:

THEOREM 1. *Let $p_1(x)$ be continuous on $[a,b]$ with $(-1)^{n-k}p_1(x) < 0$. For every $m \geq 2$ there exist $p_2(x) \in C[a,b]$ such that (4) is satisfied but $\lambda_{2,m} > \lambda_{1,m}$.*

We obtain this theorem as a consequence of the following result regarding extremal points. The ith extremal point $\theta_i(a)$ (cf. [**6**]), relative to the equation

$$y^{(n)} + p(x) y = 0 \tag{7}$$

and system (6), is defined (when it exists) as the ith value of b in (a,∞) for which there exists a nontrivial solution of (7) which satisfies (6).

Let us suppose now that $k = n-1$; in agreement with Butler and Erbe [**3**] we say that the system (6) is admissible if, having called s the unique index from $0, \ldots, n-1$ that does not belong to $\{i_1, \ldots, i_{n-1}\}$, we have $j_1 \leq s$. If we set $p(x) = p_j(x)$, $j = 1, 2$, in (7), then the corresponding ith extremal point is indicated by $\theta_{j,i}$.

THEOREM 2. *Let $p_1(x)$ and m be given, where $p_1(x)$ is continuous and positive on $[a,\infty)$, and $m \geq 1$ $[m \geq 2]$, and suppose that $\theta_{1,m}$ exists. If system (6) is not admissable [admissible] there exists $p_2(x) \in C[a,\infty)$ such that $p_2(x) \geq p_1(x) > 0$, $\theta_{2,m}$ exists, and $\theta_{2,i} > \theta_{1,i}$ for $1 \leq i \leq m$ $[2 \leq i \leq m]$.*

We remark that if (6) is admissible, then $\theta_{2,1} \leq \theta_{1,1}$ (see [**2**, Theorem 2]).

A comparison principle. Let us begin by stating some notation which we use in the following.

We say that a nonnull vector of $\mathbf{R}^n$, $\eta = (\eta_1, \ldots, \eta_n)$, has the D-property if there exist no three indices i, j, k such that $i < j < k$ and $\eta_i\eta_j < 0$; $\eta_j\eta_k < 0$.

We say that η has the strictly D-property if there exists an index i such that the real numbers $\eta_1, \dots, \eta_{i-1}, (-1)\eta_{i+1}, \dots, (-1)\eta_n$ are all different from zero and have the same sign.

If η has the D-property, we denote by $r(\eta)$ the greatest index such that $\eta_{r(\eta)} \neq 0$ and $\eta_{r(\eta)}\eta_i \geq 0$ for every $i \leq r(\eta)$.

Now let $y(x)$ be the solution of the Cauchy problem

$$(8) \qquad y^{(n)} + p(x)y = 0, \quad y^{(i)}(\xi) = \eta_{i+1}, \qquad i = 0, 1, \dots, n-1,$$

with $\xi \in \mathbf{R}$ and $p(x) > 0$.

If $\eta_i = \delta_{i,l}$ for a given l, $1 \leq l \leq n$, the solution of (8) will be denoted by $u_l(x)$. These solutions will also be called the principal solution of (8).

Every solution $y(x)$ of (8) can have only isolated zeros in a compact interval $[\xi, c], c > \xi$ (cf. [**4**, Proposition 1, p. 81]). Also, for the form of the equation and Rolle's theorem the quasi-derivatives of $y(x)$, that is $y(x), y'(x), \dots, y^{(n-1)}(x)$, can have only isolated zeros.

Let $z_1 < \dots < z_m$ be the ordered set of the zeros (eventually empty) of the quasi-derivatives of $y(x)$ in an interval $(\xi, c]$ and let $Y(x)$ be the vector $(y(x), y'(x), \dots, y^{(n-1)}(x))$.

LEMMA 1. *If η has the D-property, then $Y(x)$ has the strictly D-property for $x > \xi$. Moreover $y^{(j)}(x)$, $0 \leq j \leq n-1$, vanishes at the point z_i, $i \geq 1$, if and only if $j \equiv (r(\eta) - i) \bmod n$.*

PROOF. It is not restrictive to assume $\eta_{r(\eta)} > 0$. Let $\varepsilon > 0$ such that $0 < \varepsilon < z_1 - \xi$. In the interval $(\xi, \xi + \varepsilon)$ the functions $y^{(i)}(x)$ are all positive and increasing for $i = 0, \dots, r(\eta) - 2$; all negative and decreasing for $i = r(\eta), r(\eta) + 1, \dots, n-1$, while $y^{(r(\eta)-1)}(x)$ is positive and decreasing. This situation can change only if $y^{(r(\eta)-1)}(x)$ vanishes. Therefore, if z_1 exists, it must be a zero of $y^{(r(\eta)-1)}(x)$, moreover only this quasi-derivative of $y(x)$ vanishes at this point, and $Y(x)$ has the strictly D-property for $x \in (\xi, z_1]$. This argument can be repeated in every interval $(z_i, z_{i+1}]$, $i = 1, \dots, m-1$, proving the lemma.

Let j, $0 \leq j \leq n-1$, be a fixed index and consider the functions $u_1^{(j)}(x), u_2^{(j)}(x), \dots, u_n^{(j)}(x)$. Denote by $w_1 < \dots < w_m$ the ordered set (possibly empty) of the zeros of these functions on an interval $(\xi, c]$.

LEMMA 2. *$u_l^{(j)}(x)$, $1 \leq l \leq n$, vanishes at the point w_i, $i \geq 1$, if and only if $l \equiv (i + j) \bmod n$.*

PROOF. The functions $u_{l_1}^{(j)}(x), u_{l_2}^{(j)}(x)$, $l_1 \neq l_2$, cannot have a common zero w_i on $(\xi, c]$, otherwise there is a nontrivial linear combination $v(x)$ of them with two quasi-derivatives which vanish at w_i; since the vector $(v(\xi), v'(\xi), \dots, v^{(n-1)}(\xi))$ has the D-property, this is in contradiction to Lemma 1. Moreover between two zeros of $u_{l_1}^{(j)}(x)$ there is a zero of every function $u_l^{(j)}(x)$, $l \neq l_1$; otherwise there exists (see [**4**, Lemma 1, p. 4]) a nontrivial linear combination of two principal solutions with two quasi-derivatives which vanish at a point $x_0 > \xi$, again in contradiction to Lemma 1.

Since $u_{j+1}^{(j)}(\xi) = 1$ and $u_l^{(j)}(\xi) = 0$ for every $l \neq j+1$, from the preceding observations it follows that if w_1 exists, it must be a zero of $u_{j+1}^{(j)}(x)$.

Suppose now that the lemma is true for $w_1, \ldots, w_i$, but not for w_{i+1}. This means that w_i is a zero of $u_l^{(j)}(x)$ and, if $l < n$ [$l = n$], that w_{i+1} is a zero of $u_{l_1}^{(j)}(x)$, with $l_1 \neq l+1$ [$l_1 \neq 1$]. Since all zeros w_t, $t \leq i$, are simple, it follows that

$$u_l^{(j)}(w_{i+1})u_{l+1}^{(j)}(w_{i+1}) < 0 \qquad [u_n^{(j)}(w_{i+1})u_1^{(j)}(w_{i+1}) > 0].$$

So there exists $\alpha > 0$ [$\alpha < 0$] for which $v_1(x) = u_l(x) + \alpha u_{l+1}(x)$ [$v_1(x) = u_n(x) + \alpha u_1(x)$] is such that $v_1^{(j)}(w_{i+1}) = 0$. As $u_{l_1}^{(j)}(w_{i+1}) = 0$, there exists a nontrivial linear combination $v_2(x)$ of $v_1(x)$ and $u_{l_1}(x)$ which has two quasi-derivatives which vanish at w_{i+1}, but the vector $(v_2(\xi), \ldots, v_2^{(n-1)}(\xi))$ has the D-property and this contradicts Lemma 1.

The following proposition gives us a criterion to compare the zeros of two solutions of the Cauchy problem (8).

PROPOSITION. *Suppose that $u_l^{(j)}(x)$, $j+1 \leq l$, has m zeros, $w_1 < \cdots < w_m$, on $(\xi, c]$. If η is a vector with the D-property such that $j+1 \leq r(\eta) \leq l$ and $\eta_i \neq 0$ for at least one index $i \neq l$, then the j-derivative of the solution $y(x)$ of* (8) *has at least m zeros $z_1 < \cdots < z_m$ on (ξ, w_m) and $z_i < w_i$ for every i. Moreover if $l = r(\eta)$, $y^{(j)}(x)$ has exactly m zeros on (ξ, w_m).*

PROOF. It is not restrictive to assume $\eta_{r(\eta)} > 0$, so that $\eta_i \geq 0$ for $1 \leq i \leq r(\eta)$, $\eta_i \leq 0$ for $r(\eta)+1 \leq i \leq n$. Suppose first that $l = r(\eta)$. From Lemma 2 it follows that at the point w_i we have for all the indices $t \neq l$, either $\eta_t = 0$ or $\operatorname{sgn}[\eta_t u_t^{(j)}(w_i)] = (-1)^i$. Since $\eta_t \neq 0$ for at least an index $t \neq l$, from the relation $y^{(j)}(x) = \sum_{i=1}^n \eta_i u_i^{(j)}(x)$ and by continuity it follows that $y^{(j)}(x)$ has a zero in every interval (w_i, w_{i+1}), $i = 1, \ldots, m-1$. But $r(\eta) \geq j+1$ so that $y^{(j)}(x) > 0$ for $\xi < x < \xi + \varepsilon$ and ε sufficiently small; this implies that $y^{(j)}(x)$ must have a zero also in the interval (ξ, w_1). If $y^{(j)}(x)$ has two zeros in an interval (w_i, w_{i+1}) or (ξ, w_1), then it is possible to consider a linear combination $v(x)$ of $y(x)$ and $u_l(x)$ which has two quasi-derivatives which vanish at a point $x_0 > \xi$. Since $r(\eta) = l$, the initial conditions of $v(x)$ determine a vector with the D-property and this contradicts Lemma 1.

If $l > r(\eta)$, then by Lemma 2 $u_{r(\eta)}^{(j)}(x)$ has m zeros $w_1' < w_2' < \cdots < w_m'$ on (ξ, w_m) and $w_i' < w_i$ for every i. Now if $\eta_i \neq 0$ only for $i = r(\eta)$, then the proof is trivial; otherwise the conclusion follows from the case $l = r(\eta)$.

We consider now the particular case of problem (8) for which $\xi = 0$ and $p(x)$ is constant, that is $p(x) = k^n$, $k > 0$. The problem becomes

$$y^{(n)} + k^n y = 0, \quad y^{(i)}(0) = \eta_{i+1}, \qquad i = 0, 1, \ldots, n-1. \tag{9}$$

Since in this case we are interested in the dependence of k, we indicate the solution of (9) with $y(x,k)$ and the principal solutions with $u_l(x,k)$, $1 \leq l \leq n$.

For every $k > 0$ the principal solutions are oscillatory (see [**6**, Remark, p. 188]). If η is a vector with the D-property, then from the Proposition the solution of (9) is also oscillatory for every k. Then it is possible to consider the function $h(k)$ which associates the abscissa of the first zero of $y(x,k)$ in the interval $(0, +\infty)$ to k.

LEMMA 3. *Let η be a vector with the D-property and $y(x,k)$ be the solution of* (9). *Then*

$$\lim_{k \to +\infty} kh(k) = M > 0$$

and

$$\lim_{k\to+\infty} \frac{\partial^{(i)} y}{\partial x^{(i)}}(h(k),k) \Big/ \frac{\partial^{(i-1)} y}{\partial x^{(i-1)}} (h(k),k) = +\infty$$

for every i such that $2 \le i \le n-1$.

PROOF. From the relations

$$(10) \qquad \begin{aligned} y(x,k) &= \sum_{i=1}^{n} \eta_i u_i(x,k), \\ u_i(x,k) &= k^{1-i} u_i(kx,1), \qquad i = 1,2,\dots,n, \end{aligned}$$

if follows that

$$(11) \qquad \frac{\partial^{(j)} y}{\partial x^{(j)}}(h(k),k) = \sum_{i=1}^{n} \eta_i k^{1-i+j} \frac{\partial^{(j)} u_i}{\partial x^{(j)}}(kh(k),1).$$

For our definition, $h(k)$ is the first positive zero $y(x,k)$, therefore $kh(k)$ is the first positive zero of $y(x/k,k) = \sum_{i=1}^{n} \eta_i k^{1-i} u_i(x,1)$. Let t be the first index such that $\eta_t \neq 0$. For $k \to +\infty$, $kh(k)$ tends to the first positive zero w_1 of $u_t(x,1)$. Since $(\partial^{(j)} u_t / \partial x^{(j)})(w_1,1) \neq 0$ for $j = 1,2,\dots,n-1$ by Lemma 1, the proof of the lemma then follows by relation (11).

Proof of Theorem 2. Let system (6) be nonadmissible.

Let s be the unique index which does not belong to $\{i_1, i_2, \dots, i_{n-1}\}$; then $\theta_{l,i}(a)$, $l = 1,2$, is the ith zero of the j_1th derivative of the solution $u_{s+1}(x)$ of (8), where $p(x) = p_l(x)$ and $\xi = a$. Let x_1 be the first zero greater than a of $u_{s+1}(x)$. Since $j_1 > s$, from Lemma 1 it follows that $a < x_1 < \theta_{1,1}(a)$. We denote also by $u_*(x)$ the principal solution $u_n(x)$ of (8), where $p(x) = p_1(x)$ and $\xi = x_1$, and by $\theta_i(x_1)$ the ith zero greater than x_1 of $u_*^{(j_1)}(x)$.

Let us suppose first that $\theta_m(x_1)$ exists. By Lemma 1, $u_{s+1}^{(i)}(x_1) < 0$ for $i = 1,\dots,n-1$. Applying the Proposition with $\xi = x_1$ and $l = n$ it results that $\theta_i(x_1) > \theta_{1,i}(a)$ for $i = 1,\dots,m$. Since the zeros $\theta_i(x_1)$ are simple, by the continuous dependence of the initial conditions and the Proposition there exists $\overline{x} < x_1$ and $\delta > 0$ such that for every vector γ, $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n)$, with $\gamma_n = 1$ and $0 \le \gamma_i \le \delta$ for $i = 1,\dots,n-1$, and for every $x_0 \in [\overline{x}, x_1]$ the j_1th derivative of the solution of the problem

$$y^{(n)} + p_1(x) y = 0, \quad y^{(i)}(x_0) = \gamma_{i+1}, \qquad i = 0,\dots,n-1,$$

has exactly m zeros $z_1 < \cdots < z_m$ in $(x_0, \theta_m(x_1))$ and we have that

$$(12) \qquad \theta_{1,i}(a) < z_i, \qquad i = 1,\dots,m.$$

Now let η be the vector whose components are $\eta_i = u_{s+1}^{(i-1)}(\overline{x})$. By Lemma 3, there exists k_0 such that $h(k_0) + \overline{x} < x_1$, $k_0^n > \max\{p_1(x), x \in [a,x_1]\}$, and if $y(x,k)$ is the solution of (9) it follows that

$$(13) \qquad 0 < \frac{\partial^{(i)} y}{\partial x^{(i)}}(h(k_0),k_0) \Big/ \frac{\partial^{(n-1)} y}{\partial x^{(n-1)}} (h(k_0),k_0) < \delta, \qquad i = 1,\dots,n-2.$$

Consider the function

$$\tilde{p}(x) = \begin{cases} p_1(x) & \text{for } a \le x \le \overline{x}, \\ k_0^n & \text{for } \overline{x} < x \le \overline{x} + h(k_0), \\ p_1(x) & \text{for } x > \overline{x} + h(k_0). \end{cases}$$

For Lemma 1 the j_1th derivative of the solution $\tilde{u}_{s+1}(x)$ of (8) with $p(x) = \tilde{p}(x)$ and $\xi = a$ does not vanish in $(a, \overline{x} + h(k_0)]$; from (13) and (12) it follows then that the ith zero of $\tilde{u}_{s+1}^{(j)}(x)$ is greater than $\theta_{i,1}(a)$ for every $i \le m$. The existence of a continuous function $p_2(x) \ge \tilde{p}(x)$ which verifies the theorem then follows by the fact that the zeros of $\tilde{u}_{s+1}^{(j)}(x)$ are simple and from the classical result on differential equations.

Consider now the case for which $\theta_m(x_1)$ does not exist. Since the principal solutions of (9) are oscillatory, from (10) and Rolle's theorem it follows that the ith, $i \ge 1$, zero of $u_{s+1}^{(j_1)}(x, k)$ tend to zero for $k \to +\infty$. By Lemma 1 the vector η, whose components are $\eta_i = u_{s+1}^{(i-1)}(\theta_{1,m}(a))$, $i = 1, \ldots, n$, has the D-property, therefore for the Proposition also the ith zero of the j_1th derivative of the solution of (9) which correspond to this vector tends to zero for $k \to +\infty$. So it is possible to consider a function $p_1'(x)$ such that $p_1'(x) \ge p_1(x)$, $p_1'(x) = p_1(x)$ for $a \le x \le \theta_{1,m}(a)$, and the point $\theta_m(x_1)$ corresponding to the new function $p_1'(x)$ exists. The proof of the theorem then follows from the preceding case.

Let (6) be admissible.

By Lemma 1 the first zero x_1 of $u_{s+1}(x)$ belongs to the interval $[\theta_{1,1}(a), \theta_{1,2}(a))$. Therefore if we proceed in the same way as in the case for which system (6) is not admissible, we can prove the existence of a function $p_2(x) \ge p_1(x)$ such that $\theta_{2,i}(a) > \theta_{1,i}(a)$ for $2 \le i \le m$ and this completes the proof of the theorem.

Proof of Theorem 1. Suppose first that $k = n - 1$.

The function $p_1(x)$ can be considered to be defined on all of the interval $[a, +\infty)$ setting $p_1(x) = p_1(b)$ for $x > b$. If system (6) is admissible, then $\lambda_{1,1} > 0$ (see [**5**, Corollary 3]). Moreover $\lambda_{1,m}$ is the mth eigenvalue of problem (5), (6), where $p(x) = p_1(x)$, if and only if b is the mth extremal point relative to equation $y^{(n)} + \lambda_{1,m} p_1(x) y = 0$ and system (6) (see [**5**, Theorem 3]). By Theorem 2 there exists $p_2(x) \ge p_1(x)$ such that the mth ($m \ge 2$) extremal point relative to the equation $y^{(n)} + \lambda_{1,m} p_2(x) y = 0$ and system (6) is greater than b. Since the positive eigenvalues of (1), (2) are decreasing functions of the point b (see [**6**, Corollary 5]), the mth eigenvalue λ_m of problem (5), (6), where $p(x) = \lambda_{1,m} p_2(x)$, is greater than 1. Therefore $\lambda_m = \lambda_{2,m}/\lambda_{1,m} > 1$ and then $\lambda_{2,m} > \lambda_{1,m}$.

If the system (6) is not admissible, then $\lambda_{1,1} = 0$ and $\lambda_{1,m} > 0$ for $m \ge 2$; therefore we can prove the theorem as in the preceding case using Theorem 2.

Suppose now that $k = 1$.

We remark that $y(x)$ is a solution of problem (5), (6) if and only if the function $z(x) = y(b + a - x)$ is a solution of problem

$$z^{(n)} + (-1)^n \lambda p(b + a - x) z = 0, \tag{14}$$

$$\begin{aligned} z^{(j_1)}(a) &= \cdots = z^{(j_{n-k})}(a) = 0, \\ z^{(i_1)}(b) &= \cdots = z^{(i_k)}(b) = 0. \end{aligned} \tag{15}$$

Therefore the eigenvalues of problem (5), (6) are the same as the eigenvalues of problem (14), (15). It follows that the case $k = 1$ can be reduced to the case $k = n - 1$, and this completes the proof of the theorem.

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