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WEAK SPECTRAL SYNTHESIS

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ABSTRACT. As an approach to the union problem for sets of spectral synthesis (S-sets), the class of weak S-sets is introduced. This class contains all finite unions of S-sets, and it has many properties analogous to those of Calderón sets. It is closed under the operation of forming finite unions, but, in contrast to Calderón sets, it is not closed under countable unions.

A major open problem in commutative harmonic analysis is the union problem, the question of whether the union of two S-sets is an S-set (see Rudin [**8**, p. 172], and Graham and McGehee [**3**, p. 71]). In connection with this problem, the author considers the class of weak S-sets (see below for definition). This class is closed under the operation of forming finite unions, contains every S-set, and these sets have the property that if the boundary of a closed set is a weak S-set, so is the set. In general, the properties of weak S-sets resemble those of Calderón sets. However, unlike Calderón sets, there is a sequence of weak S-sets whose union is a closed set which is not a weak S-set.

The idea of investigating weaker forms of spectral synthesis has been discussed by Herz [**4**], Domar [**2**], and Müller [**7**], but these authors, for a class of sets associated with a given notion of weak spectral synthesis, did not examine set-theoretic properties of the class, so the present paper has an emphasis which is quite different from theirs.

The discussion is largely restricted to the context of locally compact abelian (LCA) groups which are not discrete. If G is the character group of the LCA group Γ, then $A(G)$ is the algebra of Fourier transforms of functions in the convolution algebra $L^1(\Gamma)$. When $A(G)$ is given the usual $L^1(\Gamma)$ norm (i.e. $\|f\|_{A(G)} = \|\phi\|_{L^1(\Gamma)}$ if $\hat{\phi} = f$), $A(G)$ becomes a commutative regular semisimple Banach algebra isometrically isomorphic to $L^1(\Gamma)$.

We specify the following nearly standard notation. Let E be a closed subset of G. Then $K(E) = \{f \in A(G) \colon f = 0 \text{ on } E\}$, $j(E) = \{f \in A(G)\colon f$ has compact support disjoint from $E\}$, and $J(E)$ is the closure of $j(E)$ in the $A(G)$ norm. The zero set $Z(I)$ of an ideal I in $A(G)$ is defined by $Z(I) = \bigcap\{Z(f)\colon f \in I\}$, where $Z(f)$ denotes the zero set of the function f. The smallest ideal with zero set E is $j(E)$; the smallest closed ideal with zero set E is $J(E)$, and the largest closed ideal with zero set E is $K(E)$. If $J(E) = K(E)$, then E is the zero set of a unique closed ideal, and E is a set for which spectral synthesis holds (E is an S-set). If for each f in $K(E)$, the closed ideal $I(f)$ in $A(G)$, generated by f, is the same as the closure of the ideal $f \cdot j(E)$, then E is a Calderón set, or a C-set. Every C-set is an S-set, but it is not known whether the two classes of sets coincide. Sets

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which satisfy a condition at least as strong as the C-set condition are called sets of *strong* synthesis. For harmonic analysis, see the monographs of Benedetto [**1**], Graham and McGehee [**3**], Katznelson [**5**], and Rudin [**8**]. For Banach algebras, see Graham and McGehee [**3**], Katznelson [**5**], and Stout [**9**]. For results on weak spectral synthesis, see the interesting papers by Kirsch and Müller [**6**], Domar [**2**], and Müller [**7**].

1. Weak S-sets for Banach algebras. From this point on, let A be a commutative regular semisimple Banach algebra with maximal ideal space $\Delta(A)$, Gelfand transform $x \to \hat{x}$, and let E be a closed subset of $\Delta(A)$. If $J(E) \neq K(E)$, then E is not an S-set for A, and the quotient algebra $K(E)/J(E)$ is a radical algebra. Hence, if $x \in K(E)$, $\lim_n \|x^n + J(E)\|^{1/n} = 0$, so, by definition, $\|x^n + J(E)\| = \operatorname{dist}(x^n, J(E))$ and $\lim_n \operatorname{dist}(x^n, J(E)) = 0$ for each x in $K(E)$. One way in which this limit could be zero is for each x in $K(E)$ to have a corresponding integer N (depending on x) such that $x^N \in J(E)$ (i.e. the coset of each x is nilpotent in the quotient algebra).

1.1 DEFINITION. Let E be a closed subset of $\Delta(A)$, and let $x \in K(E)$. If there exists an $n \geq 1$ such that $x^n \in J(E)$, define $n(x)$, the characteristic of x, to be the smallest n for which $x^n \in J(E)$. If no such power of x exists, define $n(x)$ to be infinite. E is a weak S-set if for each x in $K(E)$, $n(x)$ is finite; the characteristic of E, $\xi(E)$, is defined by $\xi(E) = \sup\{n(x): x \in K(E)\}$, and E is an r-set if $\xi(E) = r$. The class of S-sets is the same as the class of 1-sets, so the union problem is the question of whether a finite union of r-sets is an r-set for the case $r = 1$. A result due to Rudin [**8**, p. 174] shows that if G is an infinite compact abelian group, then there is an f in $A(G)$ such that the closed ideals $I(f^n)$, $n = 1, 2, 3, \ldots$, are all distinct. Thus, G contains a closed set $(Z(f))$ which is not a weak S-set. Varopoulos [**10**], shows that in this notation, $\xi(E) = [(n+1)/2]$ if E is a sphere in R^n, so that, by changing n, one can make the value of ξ on R^n as large as desired. One can use this result to show that in T^∞ there are weak S-sets of arbitrarily large characteristic. The following theorem answers the question of whether $\xi(E)$ can ever be infinite if E is a weak S-set.

1.2 THEOREM. *Let A be a commutative regular semisimple Banach algebra. If E is a weak S-set in $\Delta(A)$, then $\xi(E)$ is finite.*

PROOF. For each $n \geq 1$, let $S_n = \{x \in K(E): x^n \in J(E)\}$, and let $H(E) = \bigcup_{n=1}^\infty S_n$. Then $H(E) = K(E)$ by hypothesis. For each n, S_n is closed, and $\lambda x \in S_n$ if $x \in S_n$ and $\lambda \in \mathbf{C}$. By the Baire category theorem, S_m has nonempty interior for some m. Hence for some $v \in S_m$ and $\varepsilon > 0$, if $u \in K(E)$ and $\|u - v\| < \varepsilon$, then $u \in S_m$, so that $\lambda u \in S_m$ also, if $\lambda \in \mathbf{C}$. We show that $K(E) = S_m + S_m$. Let $x \in K(E)$ and let $\lambda \in \mathbf{C}$ be chosen so that $\|x/\lambda\| < \varepsilon$. Then $\|v + x/\lambda - v\| < \varepsilon$, so $v + x/\lambda \in S_m$, and therefore $\lambda v + x \in S_m$. Thus if $y = -\lambda v$ and $z = \lambda v + x$, then $x = y + z$, with $y, z \in S_m$, as desired. With this representation of x,

$$x^{2m} = (y+z)^{2m} = \sum_{k=0}^{2m} \binom{2m}{k} y^{2m-k} z^k, \quad \text{so } x^{2m} \in J(E).$$

Thus $n(x) \leq 2m$ for all x in $K(E)$, i.e. $\xi(E) \leq 2m$.

REMARK. This proof shows that if B is an arbitrary nilpotent radical algebra, then, for some N, $x^N = 0$ for all x in B.

1.3 THEOREM. *If $K(E)$ is finitely generated in A, then E is a weak S-set if and only if the characteristic of each of these generators is finite.*

PROOF. Let $K(E) = I(x_1, x_2, \dots, x_n) = \overline{x_1A + x_2A + \cdots + x_nA}$, and let m be the largest of the characteristics of these n generators, so each of these generators belongs to S_m. Then $a_1x_1 + a_2x_2 + \cdots + a_nx_n \in S_{nm}$ if $a_i \in A$, $i = 1, 2, 3, \dots, n$. Since S_{nm} is closed, $K(E) \subseteq S_{nm}$, i.e. E is a weak S-set. The converse follows from the definition of a weak S-set.

2. Weak S-sets for $A(G)$.

2.1 THEOREM. *If E is a closed subset of the nondiscrete LCA group G such that the boundary of E is an r-set, then $\xi(E) \leq r + 1$.*

PROOF. If $f \in K(E)$, there exists a sequence $\{u_k\}$ in $j(\text{bdry}\, E)$ such that $u_k \to f^r$. Hence $u_kf \to f^{r+1}$ and $u_kf \in J(E)$, so $f^{r+1} \in J(E)$.

2.2 THEOREM. *If E_1 and E_2 are weak S-sets, so is $E_1 \cup E_2$, and $\xi(E_1 \cup E_2) \leq \xi(E_1) + \xi(E_2)$.*

PROOF. Let $\xi(E_1) = m$, $\xi(E_2) = n$, and suppose that $f \in K(E_1 \cup E_2)$ and $\varepsilon > 0$ are arbitrarily chosen. Then there exists a u in $j(E_1)$ such that $\|f^m - u\| < \varepsilon/2\|f^n\|$, and a function v in $j(E_2)$ such that $\|f^n - v\| < \varepsilon/2\|u\|$. Thus $\|f^{m+n} - uv\| < \varepsilon$ and $uv \in j(E_1 \cup E_2)$. Hence $n(f) \leq m+n$ for all f in $k(E_1 \cup E_2)$, so $\xi(E_1 \cup E_2) \leq m+n$.

REMARK. This result shows that the collection of weak S-sets is closed under the operation of forming finite unions so that, in particular, finite unions of S-sets are weak S-sets.

2.3 THEOREM. *If the union and intersection of two closed sets are weak S-sets, then so also are the two closed sets.*

PROOF. Let $E = E_1 \cup E_2$ be the union, $F = E_1 \cap E_2$ be the intersection of the two sets, and let $\xi(E) = m$, $\xi(F) = n$.

If $f \in K(E_1)$, then $f \in K(F)$, and there exists a sequence $\{u_k\}$ in $j(F)$ such that $u_k \to f^n$. Thus $u_kf \to f^{n+1}$, and $E_1 \cap \text{supp}(u_kf) \cap E_2 = \varnothing$. By the normality of $A(G)$, for each k there corresponds a function $w_k \in j(E_2)$ such that $w_k = 1$ in a compact neighborhood of $E_1 \cap \text{supp}(u_kf)$, and $u_kw_kf \in K(E)$, so $(u_kw_kf)^m \in J(E)$. Since $(u_kf)^m = (u_kw_kf)^m$ locally on a neighborhod of $E_1 \cap \text{supp}(u_kf)$, $u_k^mf^m$ belongs locally to $J(E)$ at each point of E_1, and since $J(E) \subseteq J(E_1)$, $u_k^mf^m \in J(E_1)$. Hence $f^{mn} \cdot f^m = f^{mn+m} \in J(E_1)$ so that $\xi(E_1) \leq mn + m$, and similarly $\xi(E_2) \leq mn + m$.

REMARK. This theorem shows that weak S-sets can be constructed "polyhedrally" in the same way that C-sets can be (cf. Rudin [**8**, pp. 169–170] and Warner [**13**, pp. 100–101]).

2.4 COROLLARY. *If $\xi(\text{bdry}\, E) = n$, then $\xi(E) \leq n + 1$.*

PROOF. Note that $\xi(E \cup \overline{E^c}) = 1$ and $\xi(E \cap \overline{E^c}) = n$.

The following result is a corollary of work due to Varopoulos.

2.5 COROLLARY. (a) *For each $n \geq 1$, the unit sphere E in R^n is an r-set, where $r = [(n+1)/2]$ [**10**, Theorem 3, p. 379].*

(b) *For each* $n \geq 1$, T^n *contains a small sphere which is an* r*-set, with* $r = [(n+1)/2]$ [**12**, §9.1, *p.* 102].

(c) *Let* G *be an LCA group, and* H *an* S*-set in* G. *If* E *is a closed subset of* H *such that* E *is an* r*-set for* $A(H)$, *then* E *is an* r*-set for* $A(G)$ [**12**, §4.4, *p.* 76].

(d) *If* $r \geq 1$, T^∞ *contains a weak* S*-set for which* $\xi(E) = r$.

(e) *Every nonempty open set in* T^∞ *contains an* r*-set* (*for* $r = 1, 2, 3, \dots$).

PROOF. (a) Varopoulos proves that if $I = K(E)$ and I^m is the closed ideal generated by the products $f_1 f_2 f_3 \cdots f_m$, where $f_1, f_2, \dots, f_m \in I$, then $J(E) = I^r \subsetneqq I^{r-1} \subsetneqq \cdots \subsetneqq I = K(E)$. Thus $\xi(E) \leq r$, and by a lemma of Graham and McGehee [**3**, 11.2.10, p. 321] (with $k = 1$), $K(E)$ is generated by a single function f, for which $f^{r-1} \notin J(E)$, so that $\xi(E) = r$.

(d) If $m = 2r - 1$, T^m is a compact subgroup of T^∞, and by (b), T^m contains an r-set E, which is an r-set in T^∞ by (c).

(e) Every open neighborhood of 0 in T^∞ contains a sphere whose intersection with T^m contains an r-set.

2.6 THEOREM. *There is a pairwise disjoint sequence* $\{E_i\}$ *of weak* S*-sets in* T^∞ *such that the union* $E = \bigcup_{i=1}^\infty E_i$ *is a compact set which is not a weak* S*-set.*

PROOF (see Varopoulos [**11**, Chapter V, p. 18]). Let $a \in T^\infty$ and let $\{a_n\}$ be a sequence of distinct points all different from a, such that $a_n \to a$. Let $\{U_m\}$ be a decreasing sequence of compact neighborhoods of a such that $\{a\} = \bigcap_{m=1}^\infty U_m$. There exists an m_1 such that $a_1 \notin U_{m_1}$, and a compact neighborhood $V_{n_1}(a_1)$ disjoint from U_{m_1}. Since $a_n \to a$, there is an $n_2 > n_1$ such that $a_{n_2} \in U_{m_1}$, and as above, there is a disjoint pair of compact neighborhoods $U_{m_2}(a)$ (with $m_2 > m_1$) and $V_{n_2}(a_{n_2})$, both contained in U_{m_1}. By induction one obtains a pair $\{U_{m_i}\}$, $\{V_{n_i}\}$ of sequences of compact neighborhoods such that $\{V_{n_i}\}$ is a pairwise disjoint sequence, and for each $i \geq 1$, $U_{m_{i+1}}$ and $V_{n_{i+1}}$ are disjoint subsets of U_{m_i}. By Corollary 2.5(e), V_{n_i} contains a set E_i which is an i-set for each $i \geq 2$. We let $E_1 = \{a\}$, and $E = \bigcup_{i=1}^\infty E_i$, so that E is closed. Let $i > 1$ be fixed. Since $\xi(E_i) = i$, there exists an $f \in K(E_i)$ such that $f^{i-1} \notin J(E_i)$. Because E_i and $\bigcup_{k \neq i} E_k$ are compact disjoint sets, and $A(T^\infty)$ is a normal Banach algebra, there exists a function $u \in j(\bigcup_{k \neq i} E_k)$ such that $u = 1$ on a neighborhood of E_i

If $(uf)^{i-1} \in J(E_i)$, then $f^{i-1} = u^{i-1} f^{i-1}$ on a neighborhood of E_i, so $f^{i-1} \in J(E_i)$, which is a contradiction. Thus $(uf)^{i-1} \notin J(E_i)$, and $J(E) \subset J(E_i)$, so $(uf)^{i-1} \notin J(E)$. Since $uf \in K(E)$, $\xi(E) \geq i$. This holds for every $i > 1$, so E is not a weak S-set, by Theorem 1.2.

REMARK. If the appropriate minor modifications are made, the above proof applies to any LCA group G for which 2.5(e) holds.

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