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INVARIANT SUBSPACES FOR OPERATORS IN SUBALGEBRAS OF $L^\infty(\mu)$

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ABSTRACT. For each nontrivial subalgebra A of $L^\infty(\mu)$, let $A^2(\mu)$ denote the $L^2(\mu)$-closure of A and let $\mathcal{A} = A^2(\mu) \cap L^\infty(\mu)$. Then $A^2(\mu)$ has a nontrivial $\mathcal{A}$-invariant subspace.

We extend a recent invariant subspace result of Thomson [**4**]. Also, it is hoped that the Hilbert space formulation in this paper will more clearly expose some of the fundamental ideas of Thomson's ingenious argument and will display it as an extension of the preliminary strategy used by Brown [**1**].

Let A denote a subalgebra of $L^\infty(\mu)$, containing constants, where μ is a positive, finite, compactly supported Borel measure on the complex plane $\mathbf{C}$. To avoid trivialities we assume that $L^\infty(\mu)$ is infinite dimensional (i.e. μ is not a finite linear combination of point masses) and that A contains a nonconstant element. By $A^2(\mu)$ we mean the $L^2(\mu)$-closure of A. Let $\mathcal{A} = A^2(\mu) \cap L^\infty(\mu)$. Then $\mathcal{A}$ is an ω^*-closed subalgebra of $L^\infty(\mu)$, $\mathcal{A}^2(\mu) = A^2(\mu)$, and $\mathcal{A} \cdot A^2(\mu) = A^2(\mu)$ (see Conway [**2**]).

In [**4**] Thomson shows that if 1 and z belong to A, then $A^2(\mu)$ has a nontrivial $\mathcal{A}$-invariant subspace. This means that there exists a subspace K with $\{0\} \subsetneqq K \subsetneqq A^2(\mu)$ and $a \cdot K \subset K$ for all $a \in \mathcal{A}$. Thomson's result generalizes and simplifies the proof of Brown's invariant subspace theorem [**1**].

For $a \in \mathcal{A}$ denote multiplication by a acting on $A^2(\mu)$ by M_a. Then M_a is a subnormal operator and M_a commutes with M_b for any $b \in \mathcal{A}$ (see Conway [**2**] for terminology). Thus a test question for the existence of nontrivial hyperinvariant subspaces for subnormal operators is the existence of a nontrivial $\mathcal{A}$-invariant subspace of $A^2(\mu)$, when A is nontrivial.

We prove

THEOREM A. *If A is a nontrivial subalgebra of $L^\infty(\mu)$ containing constants, then $A^2(\mu)$ contains a nontrivial $\mathcal{A}$-invariant subspace.*

The proof of Theorem A requires several lemmas. The purpose of the first lemma is to replace the original invariant subspace problem with a presumably easier one.

LEMMA 1. *If there exists $w \in L^2(\mu)$ with $w \geq 1$, such that $\mathcal{A}^2(w d\mu)$ has a nontrivial $\mathcal{A}$-invariant subspace $\mathcal{M}$ with $\mathcal{M} \cap L^\infty(\mu) \neq \{0\}$, then $A^2(\mu)$ has a nontrivial $\mathcal{A}$-invariant subspace.*

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Note that for Lemma 1 to be useful, some restriction on the $\mathcal{A}$-invariant subspace of $\mathcal{A}^2(wd\mu)$ must be imposed. It is not hard to show that a sufficient condition that $\mathcal{M} \cap L^\infty(\mu) \neq \{0\}$ is that the $\mathcal{A}$-invariant subspace $\mathcal{M}$ have finite codimension in $\mathcal{L}^2(wd\mu)$.

PROOF OF LEMMA 1. First, we see that $1 \notin \mathcal{M}$, else $\mathcal{A} \cdot 1 \subset \mathcal{M}$ and $\mathcal{A}^2(wd\mu) = \mathcal{M}$, contradicting the nontriviality of $\mathcal{M}$.

Let y be the projection of 1 onto $\mathcal{M}^\perp$ and let $m \in \mathcal{M} \cap L^\infty(\mu)$ with $m \neq 0$.

Since $w \geq 1$, $\mathcal{A}^2(wd\mu) \subset \mathcal{A}^2(\mu) = A^2(\mu)$, so $m, y \in A^2(\mu)$. The hypotheses show that for $a \in \mathcal{A}$, $mya \in A^2(\mu)$. Define

$$\mathcal{K} = \operatorname{sp}\{mya \colon a \in \mathcal{A}\}^{-L^2(\mu)}.$$

Then $\{0\} \subsetneqq \mathcal{K} \leq A^2(\mu)$ and clearly $\mathcal{K}$ is $\mathcal{A}$-invariant. Now $(y/\overline{y})w \in L^2(\mu)$, where $y/\overline{y}$ is defined to be 0, wherever y vanishes.

Computing, we see that

$$\langle mya, (y/\overline{y})w\rangle_\mu = \langle ma, y\rangle_{wd\mu} = 0,$$

since $m \in \mathcal{M}$ and $\mathcal{M}$ is $\mathcal{A}$-invariant. But

$$\langle y, (y/\overline{y})w\rangle_\mu = \langle 1, y\rangle_{wd\mu} = \|y\|^2_{wd\mu} \neq 0,$$

so $\mathcal{K} \subsetneqq A^2(\mu)$. □

It remains to show how to find such a useful w. The next two lemmas are similar to well-known results about Cauchy transforms (see Garnett [**3**]).

LEMMA 2. *Let $\alpha \in L^\infty(\mu)$ and $\phi \in L^1(\mu)$. Then for each $0 < \varepsilon < 2$,*

$$|\phi(z)|/|\lambda - \alpha(z)|^{2-\varepsilon} \in L^1(\mu) \quad \text{for } m\text{-a.e. } \lambda \in \mathbf{C}.$$

(m denotes area Lebesgue measure.)

PROOF. Assume that the essential range of α and the support of μ, K, are contained in the open disc about 0 of radius R, $D_R(0)$. By Fubini we need only show that for $0 < \varepsilon < 2$ fixed,

$$I_\varepsilon \equiv \int_K |\phi(z)| \int_{D_R(0)} \frac{1}{|\lambda - \alpha(z)|^{2-\varepsilon}}\, dm(\lambda)\, d\mu(z)$$

is finite.

But for λ in $D_R(0)$ and μ-a.e. z, $|\alpha(z) - \lambda| < 2R$, so

$$I_\varepsilon \leq 2\pi \int_K |\phi(z)| \int_0^{2R} \frac{1}{r^{2-\varepsilon}} r\, dr \leq 2\pi \frac{(2R)^\varepsilon}{\varepsilon} \int_K |\phi|\, du < \infty. \quad \square$$

LEMMA 3. *Let $\alpha \in L^\infty(\mu)$ and $\phi \in L^2(\mu)$. Suppose that*

$$\int_K \frac{\phi(z)}{\lambda - \alpha(z)}\, d\mu(z) = 0 \quad \text{for } m\text{-a.e. } \lambda \text{ in } \mathbf{C}.$$

Then

$$\int_K \phi(z)\left[4R^2 - |\alpha(z) - w|^2\right] d\mu(z) = 0$$

for all $w \in D_R(0)$.

PROOF. Choosing R as in Lemma 2 and applying the proof of Lemma 2 with $\varepsilon = 1$, we have for any fixed $w \in D_R(0)$

$$0 = \int_{D_{2R}(w)} \left((\lambda - w)\int_K \frac{\phi(z)}{\lambda - \alpha(z)}\, d\mu(z)\right) dm(\lambda)$$
$$= \int_K \phi(z)\left[\int_{D_{2R(w)}} \frac{\lambda - w}{\lambda - \alpha(z)}\, dm(\lambda)\right] d\mu(z).$$

By Cauchy's theorem the bracketed integral equals

$$2\pi \int_0^{2R} \operatorname{Index}(\alpha(z), \partial D_r(w)) r\, dr$$
$$= 2\pi \int_{|\alpha(z)-w|}^{2R} r\, dr = \pi(4R^2 - |\alpha(z) - w|^2),$$

completing the proof. □

Recall that A is a subalgebra of $L^\infty(\mu)$ containing the constants.

LEMMA 4. *Let $\alpha \in L^\infty(\mu)$ and $\phi \in L^2(\mu)$. Suppose that*

$$\int_K \frac{\phi(z)}{\lambda - \alpha(z)}\, d\mu(z) = 0$$

for m-a.e. λ in D. Then $\phi \perp \alpha$ and $\phi \perp \overline{\alpha}$. Thus, if ϕ satisfies the condition above for each α in $\mathcal{A}$, then $\phi \perp \mathcal{A}$ and $\phi \perp \mathcal{A}^*$, $\mathcal{A}^* = \{\overline{a} \colon a \in \mathcal{A}\}$.

PROOF. Varying R in Lemma 3, we see that

$$0 = \int_K \phi(z) d\mu(z).$$

Thus for $w \in D_R(0)$, Lemma 3 gives

$$0 = \int_K \phi(z)\left(4R^2 - |\alpha(z)|^2 + 2\operatorname{Re}(w\overline{\alpha(z)}) - |w|^2\right) d\mu(z)$$
$$= -\int_K \phi(z)|\alpha(z)|^2\, d\mu(z) + 2\int_K \phi(z)\operatorname{Re}(w\overline{\alpha(z)})\, d\mu(z).$$

Varying w we get

$$0 = \int_K \phi(z)\operatorname{Re}(\alpha(z))\, d\mu(z) \quad \text{and} \quad 0 = \int_K \phi(z)\operatorname{Im}(\alpha(z))\, d\mu(z). \quad \square$$

We thank the reviewer for his suggestions on improving Lemma 4.

COROLLARY 1. *Suppose that for each $\overline{\phi} \in L^2(\mu) \ominus A^2(\mu)$ and every $\alpha \in \mathcal{A}$, we have*

$$\int_K \frac{\phi(z)}{\lambda - \alpha(z)}\, d\mu(z) = 0 \quad \textit{for m-a.e. } \lambda \in \mathbf{C}.$$

Then $\mathcal{A}$ acting on $A^2(\mu)$ is an abelian von Neumann algebra.

PROOF. Just note that using Lemma 4, the assumptions we are making force $A^2(\mu)$ to contain $\mathcal{A}^*$. Hence $\mathcal{A} = A^2(\mu) \cap L^\infty(\mu) \supset \mathcal{A}^*$ and $\mathcal{A}$ is selfadjoint on $A^2(\mu)$.

This follows since for $a \in \mathcal{A}$, M_a^* is the projection onto $A^2(\mu)$ of multiplication by $\bar{a}$. Because $\mathcal{A}$ is ω^*-closed in $L^\infty(\mu)$ it is easy to see that $\mathcal{A}$ is weak operator closed, considered first as multiplication operators on $L^2(\mu)$ and second when restricted to the $\mathcal{A}$-reducing subspace of $L^2(\mu), A^2(\mu)$.

COROLLARY 2. *If A is nontrivial and the hypotheses of Corollary* 1 *are in force, then $A^2(\mu)$ has a nontrivial $\mathcal{A}$-invariant subspace containing a bounded element*; *in fact an idempotent.*

PROOF. Using Corollary 1 we know that $\mathcal{A}$ is an abelian von Neumann algebra. But by standard results (Conway [**2**, p. 88]), $\mathcal{A}$ is generated by a single selfadjoint element α in $\mathcal{A}$. Since A is nontrivial, any nontrivial spectral projection of α generates the desired $\mathcal{A}$-invariant subspace of $A^2(\mu)$. □

We are now ready to prove Theorem A.

PROOF. By Lemma 1 we need to find a $w \geq 1$ in $L^2(\mu)$, so that $\mathcal{A}^2(wd\mu)$ contains a nontrivial $\mathcal{A}$-invariant subspace, $\mathcal{M}$, with $\mathcal{M} \cap L^\infty(\mu) \neq \{0\}$. We have two cases.

If Corollary 1 applies, then take $w \equiv 1$ and Corollary 2 gives the desired $\mathcal{A}$-invariant subspace (which is in fact reducing) with nontrivial idempotent as the bounded element.

Else there is a $\overline{\phi} \in L^2(\mu) \ominus A^2(\mu)$, an $\alpha \in \mathcal{A}$, and a $\lambda \in \mathbf{C}$ for which

$$\int_K \left(\frac{|\phi|}{|\lambda - \alpha|}\right)^{3/2} d\mu < \infty \tag{1}$$

and

$$\int_K \frac{\phi}{\lambda - \alpha} d\mu \neq 0. \tag{2}$$

(Note that since $\overline{\phi} \perp 1$, using (2) we see that α cannot be a constant.)

Let

$$w(z) = \left(1 + \frac{|\phi(z)|}{|\lambda - \alpha(z)|}\right)^{1/2}.$$

Then $w \geq 1$ and by (1) $w \in L^2(\mu)$. Define a linear functional on $\mathcal{A}$ by

$$L(b) = \int_K b \frac{\phi}{\lambda - \alpha} d\mu \quad \text{for } b \in \mathcal{A}.$$

Then by Cauchy-Schwarz and (1)

$$\begin{aligned} |L(b)| &\leq \int_K |b| \left(1 + \frac{|\phi|}{|\lambda - \alpha|}\right) d\mu \\ &\leq \|b\|_{2,wd\mu} \left(\int_K \left(1 + \frac{|\phi|}{|\lambda - \alpha|}\right)^{3/2} d\mu\right)^{1/2} \end{aligned}$$

so L defines a bounded linear functional on $\mathcal{A}^2(wd\mu)$. This functional is represented by a unique k_λ in $\mathcal{A}^2(wd\mu)$. Since $L(1) \neq 0$ by (2), k_λ is not 0.

Let

$$\mathcal{M} = \{(\lambda - \alpha)b \colon b \in \mathcal{A}\}^{-L^2(wd\mu)}.$$

Then $\{0\} \subsetneqq \mathcal{M} \leq \mathcal{A}^2(w d\mu)$ and $\mathcal{M}$ is $\mathcal{A}$-invariant. Moreover, for $b \in \mathcal{A}$ the choice of ϕ gives

$$\langle (\lambda - \alpha)b, k_\lambda \rangle_{w d\mu} = L((\lambda - \alpha)b) = \int_K (\lambda - \alpha) b \frac{\phi}{\lambda - \alpha}\, d\mu = 0.$$

Thus $\mathcal{M} \subsetneqq \mathcal{A}^2(w d\mu)$ and $\lambda - \alpha \in \mathcal{M} \cap L^\infty(\mu)$. $\square$

In conclusion we emphasize the strategy of first replacing the original invariant subspace problem by an invariant subspace problem on $P^\infty(\mu)$ (Brown [**1**]), in $P^3(\mu)$ (Thomson [**4**]), or in $P^2(w d\mu)$ (for appropriate w) as above. Of course much work may still remain to solve the original problem (!), but it would seem that an abstract operator theoretic technique is lurking in this preliminary step. To be a little more specific we believe that there is an abstract operator theoretic lemma which generalizes Lemma 1.

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