PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 2, February 1987

CONTINUITY OF DERIVATIONS ON SOME SEMIPRIME BANACH ALGEBRA

RAMESH V. GARIMELLA

ABSTRACT. If every prime ideal is closed in a commutative semiprime Banach algebra with unit, then every derivation on it is continuous. Also if derivations are continuous on integral domains, then they are continuous on semiprime Banach algebras.

1. Introduction. In [**9**] Singer and Wermer proved that the range of a continuous derivation on a commutative Banach algebra is contained in the Jacobson radical. They conjectured that the assumption of continuity is unnecessary. In [**7**] Johnson proved that if A is a semisimple Banach algebra, then every derivation on A is continuous and hence by the Singer-Wermer theorem it is zero.

In this note in §3, we prove that if A is a semiprime Banach algebra in which every prime ideal is closed, then every derivation on A is continuous. We also prove that if derivations are continuous on integral domains, then they are continuous on semiprime Banach algebras. In §2, we prove that if $\bigcap_{n\geq 1} R^n$ is contained in every closed prime ideal, then the separating ideal of every derivation is nilpotent, where R is the Jacobson radical of A. This improves a result of [**8**]. We also note that if the Jacobson radical R of A is an integral domain and if there is a nonzero element $a \in R$ such that $\bigcap_{n\geq 1} a^n R = \{0\}$, then every derivation on A is continuous, which generalizes a result of [**8**].

Throughout the following we suppose A is a commutative Banach algebra with unit. R and N will denote, respectively, the Jacobson and nil radicals of A. N is also called the prime radical of A and it consists of all the nilpotent elements of A. For any derivation D on A, let

$$S(D) = \{x \in A\colon \text{there are } x_n \to 0 \text{ with } Dx_n \to X\}$$

be the separating ideal of D. Let

$$A(S(D)) = \{x \in A\colon xS(D) = 0\}$$

be the annihilator ideal of $S(D)$. By the closed graph theorem one can see that D is continuous if and only if $S(D) = \{0\}$.

2. Even though the following is implicit in [**8**], we state and prove it as a separate lemma.

LEMMA 2.1. *$S(D)$ is nilpotent if and only if $S(D) \cap R$ is nilpotent.*

PROOF. One half of the proof is obvious. For the other half suppose that $S(D)\cap R$ is nilpotent. Then by Theorem (1) of [**8**], it follows that $D(A)$ is contained in R. Since R is a closed ideal, it follows that $S(D) \subseteq \overline{D(A)} \subseteq R$. Q.E.D.

Received by the editors October 16, 1985.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 46J05.

REMARK 2.1. If A has no nonmaximal closed prime ideals, then by Theorem 2.7 of [**3**], every separating ideal of a derivation is a nilpotent ideal. Thus if necessary we can always assume that A has nonmaximal closed prime ideals.

THEOREM 2.1. *If $\bigcap_{n\geq 1} R^n$ is contained in every closed prime ideal of A, then $S(D)$ is a nilpotent ideal.*

PROOF. By Remark 2.1, we may assume that A has nonmaximal closed prime ideals. By Lemma 2.1, it is enough to prove that $S(D)\cap R$ is nilpotent. Let $x\in S(D)\cap R$. Since $S(D)$ is a separating ideal, there is a positive integer m such that $\overline{x^m S(D)} = \overline{x^n S(D)}$ for all $n \geq m$. It follows from the Mittag-Leffler Theorem [**1**, Theorem 3.3] and the hypothesis that

$$\overline{x^m S(D)} = \bigcap_{n\geq 1} \overline{x^n S(D)} = \overline{\bigcap_{n\geq 1} x^n S(D)} \subseteq \overline{\bigcap_{n\geq 1} R^n} \subseteq P$$

for every closed prime ideal P. Thus x^{m+1} belongs to every closed prime ideal P and hence so does x. Thus $S(D)\cap R$ is contained in $S(D)\bigcap P$ for every closed prime ideal P. But as noted in the proof of [**3**, Corollary 2.4], $S(D)\cap N = \bigcap(S(D)\cap P)$, where the intersection runs over all the minimal prime ideals P (which are closed) such that $S(D)\not\subseteq P$. Hence by the above argument it follows that $S(D)\cap N = S(D)\cap R$. Thus $S(D)\cap R$ is a closed nil ideal. Therefore $S(D)\cap R$ is a nilpotent ideal (cf. [**5**]). Q.E.D.

The following result is Lemma 3 of [**8**].

COROLLARY 2.1. *If $\bigcap_{n\geq 1} R^n = \{0\}$, then $S(D)$ is nilpotent.*

REMARK 2.2. Let D be a discontinuous derivation on A. Then by Theorem 2.7 of [**1**], it follows that there is an element $a\in A$ such that $D_0 = aD$ is a discontinuous derivation on A with the following property:

$$(*)\qquad \begin{cases} \text{(i)} & A(S(D_0)) \text{ is a closed prime ideal of } A. \\ \text{(ii)} & \text{For every } x\in A, \text{ either } x\in A(S(D_0)) \text{ or } \overline{xS(D_0)} = S(D_0). \end{cases}$$

THEOREM 2.2. *If the Jacobson radical R is an integral domain and if there is an element $0\neq a\in R$ such that $\bigcap_{n\geq 1} a^n R = \{0\}$, then every derivation on A is continuous.*

PROOF. Suppose the theorem is false. Then by Remark 2.2 we may assume that there is a discontinuous derivation D satisfying $(*)$, so that $A(S(D))$ is a prime ideal. First we prove that $S(D)\subseteq R$. Since D is discontinuous by Lemma 2.1, $S(D)\cap R\neq\{0\}$. Let $0\neq s\in S(D)\cap R$. Since R is an integral domain, $s\notin A(S(D))$. Therefore by $(*)$, $\overline{s(S(D))} = S(D)$. Since R is a closed ideal, it follows that $S(D)\subseteq R$. Note that $S(D)$ is not contained in $A(S(D))$ and hence $R\not\subseteq A(S(D))$. Now consider $0\neq a$ in R such that $\bigcap_{n\geq 1} a^n R = \{0\}$. Note that $a\notin A(S(D))$. (For if $a\in A(S(D))$, then $aS(D) = \{0\}$. Since R is an integral domain it follows that $a=0$.) Therefore by $(*)$, $\overline{aS(D)} = S(D)$. Let $x\in S(D)\backslash A(S(D))$. Since $A(S(D))$ is a prime ideal, $ax\notin A(S(D))$. Therefore again by $(*)$ we get $x\in\overline{(axS(D))}$. Hence $x\in\overline{axR}$. Therefore by the equivalence conditions of class (iv) [**4**, p. 59], it follows that $\bigcap_{n\geq 1} a^n R\neq\{0\}$ which is a contradiction. Q.E.D.

The following result was proved in [**8**].

COROLLARY 2.2. *If $\bigcap_{n\geq 1} R^n = \{0\}$ and R is an integral domain, then every derivation on A is continuous.*

REMARK 2.3. Using the Mittag-Leffler Theorem [**1**, Theorem 3.3] and the same argument as that of Theorem 2.2, we can prove the following

THEOREM 2.3. *Suppose A is a commutative Banach algebra with unit which is also an integral domain. Further assume there is a nonzero closed ideal I such that $\bigcap_{n\geq 1} I^n = \{0\}$. Then every derivation on A is continuous.*

3. Following Khosravi [**8**], for any ideal I of A, we define

$$K(I) = \{x \in I\colon D^n x \in I \text{ for all } n \geq 1\},$$

where D is any derivation on A.

The following lemma can be proved very easily and so we shall omit the proof.

LEMMA 3.1. *For any ideal I, $K(I)$ is an ideal. Further $K(I)$ is a prime ideal if I is a prime ideal.*

REMARK 3.1. Let A be a commutative semiprime Banach algebra with unit. Let $D\colon A \to A$ be a discontinuous derivation on A. Since A is semiprime, as noted in the proof of Corollary 2.4 of [**3**], we get that $\bigcap(S(D) \cap P) = \{0\}$, where the instersection runs over all the minimal prime ideals P such that $S(D) \not\subseteq P$. Now again by Theorem 2.7 of [**1**], there exists an element $a \in A$ such that $D_0 = aD$ is a discontinuous derivation on A satisfying $(*)$ of Remark 2.2. Since A is a semiprime Banach algebra, $S(D_0) \not\subseteq A(S(D_0))$. Let P be a minimal prime ideal such that $S(D) \not\subseteq P$. Since $S(D_0) \cdot A(S(D_0)) = \{0\}$, we get that either $S(D_0) \subseteq P$ or $A(S(D_0)) \subseteq P$. If $S(D_0)$ is contained in every such P, then since $S(D_0) \subseteq S(D)$, we get that $S(D_0) = \{0\}$, which is false. Therefore there exists a minimal prime ideal P such that $S(D_0) \not\subseteq P$. Therefore $A(S(D_0) \subseteq P$. Since $A(S(D_0))$ is a prime ideal, we get $P = A(S(D_0))$. Thus $A(S(D_0))$ is a minimal prime ideal. Thus if necessary we can assume that there exists a discontinuous derivation D such that $A(S(D))$ is a minimal prime ideal and for every $x \in A$ either $x \in A(S(D))$ or $\overline{xS(D)} = S(D)$.

THEOREM 3.1. *The following conditions are equivalent.*

(i) *Every derivation on a commutative Banach algebra with unit which is an integral domain, is continuous.*

(ii) *Every derivation on a commutative semiprime Banach algebra with unit is continuous.*

PROOF. Obviously (ii) implies (i). So assume (i). Suppose (ii) is false. Then by Remark 3.1, there exists a semiprime Banach algebra A with unit, and a discontinuus derivation D satisfying $(*)$ of Remark 2.2 and further $A(S(D))$ is a minimal prime ideal. Now by Lemma 3.1, $K(A(S(D)))$ is a prime ideal of A contained in $A(S(D))$. Since $A(S(D))$ is a minimal prime ideal we get that $A(S(D)) = K(A(S(D)))$. Therefore $A(S(D))$ is invariant under D. Hence $\tilde{D}\colon A/A(S(D)) \to A/A(S(D))$ defined by $\tilde{D}(x + A(S(D))) = Dx + A(S(D))$ is a derivation. Since $A/A(S(D))$ is an integral domain, by hypothesis, we get that $\tilde{D}$ is continuous. Hence $S(\tilde{D}) = \{0\}$. This implies that $S(D) \subseteq A(S(D))$. Since A is a semiprime Banach algebra we get that $S(D) = \{0\}$ which is a contradiction. This completes the proof of the theorem.

THEOREM 3.2. *Let A be a semiprime commutative Banach algebra with unit in which every prime ideal is closed. Then every derivation on A is continuous.*

PROOF. Suppose the theorem is false. We may assume that there is a discontinuous derivation D satisfying the conditions mentioned in Remark 3.1.

Thus $A(S(D))$ is a minimal prime ideal and hence $K(A(S(D))) = A(S(D))$. In particular $A(S(D))$ is invariant under D and we may lift D to the integral domain $A/A(S(D))$. Note that the prime ideals of $A/A(S(D))$ are all closed. If the lifted derivation is discontinuous we may again assume that $(*)$ holds. Note that since we may now assume that A is an integral domain, $A(S(D)) = \{0\}$. Hence, if I is a closed nonzero ideal and $0 \neq x \in I$, then $\overline{xS(D)} = S(D) \subseteq I$. Thus $S(D)$ is contained in every nonzero prime ideal. Also $S(D) \subseteq R$.

If I is a nonzero ideal and if $S(x) := \{x, x^2, \ldots\}$ where $x \in S(D)$, then if $I \cap S(x) = \emptyset$, we can find a prime ideal $P \supseteq I$ with $S(x) \cap P = \emptyset$. But P is a closed ideal and this contradicts the previous observation. Hence $S(x) \cap I \neq \emptyset$ for every nonzero ideal I of A.

Now since $\overline{xS(D)} = S(D)$ for every $x \neq 0$, the argument of Theorem 8.1 of [**4**] applies and produces the existence of two elements $b, c \in S(D)$ such that $b^n \notin cA$ and $c^n \notin bA$, $n = 1, 2, \ldots$. This means $S(b) \cap cA = \emptyset$ and $S(c) \cap bA = \emptyset$. This contradicts the previous paragraph and proves that the lifted derivation must be continuous. Hence for the original derivation D, we obtain $S(D) \subseteq A(S(D))$, or $S(D)^2 = \{0\}$. But A is semiprime, so $S(D) = \{0\}$. This is a contradiction. Q.E.D.

FINAL REMARK. Banach algebras satisfying the hypothesis of Theorem 3.3 occur in the work of Sandy Grabiner [**6**] (in particular refer to Theorem 3.8, p. 176 and Theorem 3.15, p. 179). This was pointed out to me by the referee and also by Professor K. B. Laursen. I am very grateful to them. Finally, it is interesting to find some more examples of Banach algebras in which prime ideals are closed.

ACKNOWLEDGMENTS. I am very thankful to the referee for his valuable comments and suggestions in improving some proofs. Also I thank Dr. N. V. Rao for his encouragement.

REFERENCES

1. W. G. Bade and P. C. Curtis, Jr., *Prime ideals and automatic continuity for Banach algebras*, J. Funct. Anal. **29** (1978), 88–103.
2. Philip C. Curtis, Jr., *Derivations on commutative Banach algebras* (Proc., Long Beach, 1981), Lecture Notes in Math., vol. 975, Springer-Verlag, pp. 328–333.
3. J. Cusack, *Automatic continuity and topologically simple radical Banach algebras*, J. London Math. Soc. **21** (1977), 493–500.
4. J. Esterle, *Elements for a classification of commutative radical Banach algebras* (Proc., Long Beach, 1981), Lecture Notes in Math., vol. 975, Springer-Verlag, pp. 4–65.
5. S. Grabiner, *The nilpotency of Banach nil algebras*, Proc. Amer. Math. Soc. **21** (1969), 510.
6. ——, *A formal power series operational calculus for quasinilpotent operators*. II, J. Math. Anal. Appl. **43** (1973), 170–192.
7. B. E. Johnson, *Continuity of derivations on commutative algebras*,, Amer. J. Math. **91** (1969), 1–10.
8. A. Khosravi, *Derivations on commutative Banach algebras*, Proc. Amer. Math. Soc. **84** (1982), 60–64.
9. M. Singer and J. Wermer, *Derivatives on commutative normed algebras*, Math. Ann. **129** (1955), 260–264.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF TOLEDO, TOLEDO, OHIO 43606

Current address: Department of Mathematics and Statistics, Northwest Missouri State University, Maryville, Missouri 64468