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CONCERNING POLYNOMIALS ON THE UNIT INTERVAL

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ABSTRACT. Let $\mathcal{P}_n$ be the normed linear space of all polynomials p of degree $\leq n$ such that $p(1) = 0$ and $\|p\| = (\int_{-1}^{1} |p(x)|^2\, dx)^{1/2}$. We determine sharp upper bounds for $|a_n|/\|p\|$ and $|a_{n-1}|/\|p\|$ as $p(x) := \sum_{\nu=0}^{n} a_\nu x^\nu$ varies in $\mathcal{P}_n$.

According to a classical result of Chebyshev if $p_n(x) := \sum_{\nu=0}^{n} a_\nu x^\nu$ is a polynomial of degree n, then

$$|a_n| \leq 2^{n-1} \max_{-1\leq x\leq 1} |p_n(x)|. \tag{1}$$

It is also known [**1**] that

$$|a_n| \leq \frac{1\cdot 3\cdot 5\cdots(2n-1)}{n!}\left(\frac{2n+1}{2}\right)^{1/2}\left(\int_{-1}^{1}|p_n(x)|^2\,dx\right)^{1/2}. \tag{2}$$

In (1) equality holds for the Chebyshev polynomial $T_n(x) := \cos n(\arccos x)$ whereas in (2) it holds for the Legendre polynomial

$$P_n(x) := \sum_{\nu=0}^{[n/2]} \frac{(-1)^\nu (2n-2\nu)!}{2^n \nu!(n-\nu)!(n-2\nu)!} x^{n-2\nu}.$$

It was shown by Schur [**2**, Theorem III*] that if p_n vanishes at one of the endpoints -1 or 1, then (1) can be replaced by

$$|a_n| \leq 2^{n-1}\left(\cos\frac{\pi}{4n}\right)^{2n} \max_{-1\leq x\leq 1}|p_n(x)|. \tag{3}$$

Here we obtain the corresponding improvement in (2). In fact, we prove

THEOREM. *If $p_n(x) := \sum_{\nu=0}^{n} a_\nu x^\nu$ is a polynomial of degree n such that $p_n(1) = 0$, then*

$$|a_n| \leq \frac{n}{n+1}\frac{(2n)!}{2^n(n!)^2}\left(\frac{2n+1}{2}\right)^{1/2}\left(\int_{-1}^{1}|p_n(x)|^2\,dx\right)^{1/2}. \tag{4}$$

The inequality is sharp and equality holds for

$$p_n(x) := P_n(x) - \frac{1}{n^2}\sum_{\nu=0}^{n-1}(2\nu+1)P_\nu(x),$$

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where P_ν is the Legendre polynomial of degree ν with the normalization $P_\nu(1) = 1$. Besides,

$$(5) \qquad |a_{n-1}| \le \frac{(n^2+2)^{1/2}}{n+1} \frac{(2n-2)!}{2^{n-1}((n-1)!)^2} \left(\frac{2n-1}{2}\right)^{1/2} \left(\int_{-1}^{1} |p_n(x)|^2\,dx\right)^{1/2}$$

which is again sharp, as the following example shows:

$$p_n(x) := \frac{2n+1}{n^2+2} P_n(x) - P_{n-1}(x) + \frac{1}{n^2+2} \sum_{\nu=0}^{n-2} (2\nu+1) P_\nu(x).$$

In the absence of the hypothesis $p_n(1) = 0$ the factor $(n^2+2)^{1/2}/(n+1)$ appearing on the right-hand side of (5) is to be dropped [**1**, (3)].

PROOF OF THE THEOREM. Let

(6)

$$\varphi_\nu(x) := \left(\frac{2\nu+1}{2}\right)^{1/2} P_\nu(x) := \left(\frac{2\nu+1}{2}\right)^{1/2} \sum_{j=0}^{[\nu/2]} \frac{(-1)^j (2\nu-2j)!}{2^\nu j!(\nu-j)!(\nu-2j)!} x^{\nu-2j}.$$

Then

$$\int_{-1}^{1} \varphi_\nu(x)\varphi_\mu(x)\,dx = \begin{cases} 0 & \text{if } \mu \ne \nu, \\ 1 & \text{if } \mu = \nu, \end{cases}$$

and the polynomial $p_n(x)$ can be expressed uniquely in the form

$$(7) \qquad p_n(x) = \sum_{\nu=0}^{n} \alpha_\nu \varphi_\nu(x),$$

where

$$\sum_{\nu=0}^{n} |\alpha_\nu|^2 = \int_{-1}^{1} |p_n(x)|^2\,dx.$$

From (7) in conjunction with (6) it follows that

(8)

$$a_n = \left(\frac{2n+1}{2}\right)^{1/2} \frac{(2n)!}{2^n (n!)^2} \alpha_n, \qquad a_{n-1} = \left(\frac{2n-1}{2}\right)^{1/2} \frac{(2n-2)!}{2^{n-1}((n-1)!)^2} \alpha_{n-1}.$$

Now we wish to prove that if $\gamma_\mu > \gamma_\nu \ge 0$ for $\nu = 0, 1, \ldots, \mu-1, \mu+1, \ldots, n$ then under the hypothesis of the theorem

$$(9) \qquad \sum_{\nu=0}^{n} \gamma_\nu |\alpha_\nu|^2 \le (\gamma_\mu - \gamma) \sum_{\nu=0}^{n} |\alpha_\nu|^2,$$

where γ is the unique root of the equation

$$(10) \qquad \sum_{\nu=0}^{n} \frac{2\nu+1}{\gamma_\mu - \gamma_\nu - x} = 0$$

in $(0, \Gamma := \min_{0 \le \nu \le n; \nu \ne \mu} (\gamma_\mu - \gamma_\nu))$.

We write the left-hand side of (9) as

$$\begin{aligned} \sum_{\nu=0}^{n} \gamma_\nu |\alpha_\nu|^2 &= \gamma_\mu \sum_{\nu=0}^{n} |\alpha_\nu|^2 - \sum_{\nu=0; \nu\ne\mu}^{n} (\gamma_\mu - \gamma_\nu) |\alpha_\nu|^2 \\ &= \gamma_\mu \sum_{\nu=0}^{n} |\alpha_\nu|^2 - \sum_{\nu=0; \nu\ne\mu}^{n} (\gamma_\mu - \gamma_\nu - \gamma) |\alpha_\nu|^2 - \gamma \sum_{\nu=0; \nu\ne\mu}^{n} |\alpha_\nu|^2, \end{aligned}$$

where, for the moment, γ is a constant in $(0, \Gamma)$. From the hypothesis $p_n(1) = 0$ and Schwarz' inequality we obtain

$$\left|\left(\frac{2\mu+1}{2}\right)^{1/2} \alpha_\mu\right|^2 = \left|\sum_{\nu=0;\nu\neq\mu}^{n} \left(\frac{2\nu+1}{2}\right)^{1/2} \alpha_\nu\right|^2 \leq \left\{\sum_{\nu=0;\nu\neq\mu}^{n} \left(\frac{2\nu+1}{2}\right)^{1/2} |\alpha_\nu|\right\}^2$$
$$= \left\{\sum_{\nu=0;\nu\neq\mu}^{n} (\gamma_\mu - \gamma_\nu - \gamma)^{1/2}|\alpha_\nu| \left(\frac{2\nu+1}{2}\right)^{1/2} (\gamma_\mu - \gamma_\nu - \gamma)^{-1/2}\right\}^2$$
$$\leq \sum_{\nu=0;\nu\neq\mu}^{n} (\gamma_\mu - \gamma_\nu - \gamma)|\alpha_\nu|^2 \sum_{\nu=0;\nu\neq\mu}^{n} \frac{2\nu+1}{2}(\gamma_\mu - \gamma_\nu - \gamma)^{-1},$$

so that

$$-\sum_{\nu=0;\nu\neq\mu}^{n} (\gamma_\mu - \gamma_\nu - \gamma)|\alpha_\nu|^2$$
$$\leq -\frac{2\mu+1}{2}|\alpha_\mu|^2 \left\{\sum_{\nu=0;\nu\neq\mu}^{n} \frac{2\nu+1}{2}(\gamma_\mu - \gamma_\nu - \gamma)^{-1}\right\}^{-1}.$$

Now if γ happens to be the root of the equation (10) lying in $(0, \Gamma)$, then

$$\left\{\sum_{\nu=0;\nu\neq\mu}^{n} \frac{2\nu+1}{2}(\gamma_\mu - \gamma_\nu - \gamma)^{-1}\right\}^{-1} = \frac{2}{2\mu+1}\gamma$$

and we get

$$\sum_{\nu=0}^{n} \gamma_\nu|\alpha_\nu|^2 \leq \gamma_\mu \sum_{\nu=0}^{n} |\alpha_\mu|^2 - \gamma|\alpha_\mu|^2 - \gamma \sum_{\nu=0;\nu\neq\mu}^{n} |\alpha_\nu|^2 = (\gamma_\mu - \gamma) \sum_{\nu=0}^{n} |\alpha_\nu|^2$$

which proves (9).

If $\gamma_n = 1$ and $\gamma_\nu = 0$ for $\nu = 0, 1, \ldots, n-1$, then γ turns out to be equal to $(2n+1)/(n+1)^2$ and (9) reduces to

$$|\alpha_n| \leq \frac{n}{n+1} \left(\sum_{\nu=0}^{n} |\alpha_\nu|^2\right)^{1/2}. \tag{11}$$

Similarly, choosing $\gamma_{n-1} = 1$ and $\gamma_\nu = 0$ for $\nu = 0, 1, \ldots, n-2, n$, we obtain

$$|\alpha_{n-1}| \leq \frac{(n^2+2)^{1/2}}{n+1} \left(\sum_{\nu=0}^{n} |\alpha_\nu|^2\right)^{1/2}. \tag{12}$$

Combining (11), (12) with (8) we readily obtain (4), (5) respectively.

Both the inequalities (4), (5) are sharp and in each case the extremal polynomials are easily identified.

References

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