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A GENERALIZATION OF LYAPOUNOV'S CONVEXITY THEOREM TO MEASURES WITH ATOMS

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ABSTRACT. The distance from the convex hull of the range of an n-dimensional vector-valued measure to the range of that measure is no more than $\alpha n/2$, where α is the largest (one-dimensional) mass of the atoms of the measure. The case $\alpha = 0$ yields Lyapounov's Convexity Theorem; applications are given to the bisection problem and to the bang-bang principle of optimal control theory.

1. Introduction. The celebrated Convexity Theorem of Lyapounov [**8**] states that the range of a nonatomic finite-dimensional vector-valued measure is compact and convex (where throughout this paper "measure" means "countably-additive nonnegative finite measure"). The range may not be convex if the measure has atoms (see for example, Figure 1) but, as is the main purpose of this paper to prove (Theorem 1.2), a fairly sharp bound can be given on how far from convex the range can be, as a function of the mass of the largest atom. Intuitively, if the atoms all have very small mass, the range is very close to being convex.

Throughout this paper, $(X, \mathcal{F})$ will denote a measurable space; $\mathcal{M}_n$ is the set of n-dimensional measures on $(X, \mathcal{F})$ (i.e., $\mathcal{M}_n = \{(\mu_1, \ldots, \mu_n) \colon \mu_i$ is a measure on $(X, \mathcal{F})$ for all $i \leq n\}$; and $R(\vec{\mu})$ is the range of $\vec{\mu} = (\mu_1, \ldots, \mu_n) \in \mathcal{M}_n$. The first theorem, a result of Lyapounov [**8**], states that the range of *every* vector measure (nonatomic or not) is always compact; this conclusion is classically proved in conjunction with the Convexity Theorem (see for example Diestel and Uhl [**3**], Halmos [**4**], Lindenstrauss [**7**], or Lyapounov [**8**]). However, only the conclusion in the case $n = 1$ will be used in the proof of Theorem 1.2, and this case is fairly easy to establish directly without convexity (cf. Halmos [**5**, Problem 4, p. 174]).

THEOREM 1.1 (LYAPOUNOV [**8**]). *If $\vec{\mu} \in \mathcal{M}_n$, then $R(\vec{\mu})$ is compact.*

To state the next theorem, the main result of this paper, some additional notation and definitions are needed: $\operatorname{co}(A)$ is the convex hull of $A \subset \mathbf{R}^n$; $\|x\|$ is the Euclidean norm of $x \in \mathbf{R}^n$; $d(x, y) = \|x - y\|$ is the distance between x and y; and $d(x, A) = \inf\{d(x, y) \colon y \in A\}$ is the distance from x to the set A.

DEFINITION. For $A \subset \mathbf{R}^n$, $D(A) = \sup\{d(x, A) \colon x \in \operatorname{co}(A)\}$. (For a set A, $D(A)$ represents the maximum "dent size" of A; see Figure 1.)

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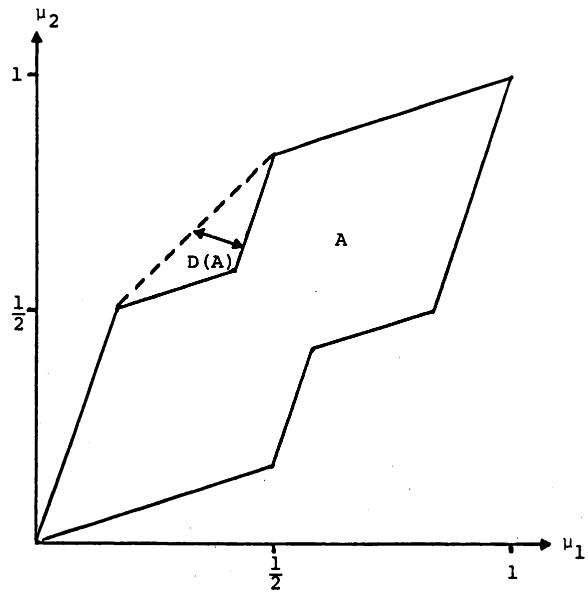

FIGURE 1

The set A is the range of the vector Borel measure (μ_1, μ_2) on $[0,1]$ defined by: $\mu(\{0\}) = \mu_2(\{0\}) = \frac{1}{3}$; $\mu_1 = \lambda$ and $\mu_2 = \frac{1}{3}\lambda$ on $(0, \frac{1}{2}]$; and $\mu_1 = \frac{1}{3}\lambda$, $\mu_2 = \lambda$ on $(\frac{1}{2}, 1]$ (where $\lambda =$ Lebesgue measure).

DEFINITION. $\mathcal{P}_n(\alpha) = \{\vec{\mu} \in \mathcal{M}_n : \mu_i(A) \leq \alpha$ for all $i \leq n$ and all μ_i-atoms $A\}$. (So $\mathcal{P}_n(\alpha)$ is the collection of n-dimensional vector measures none of whose coordinate measures have atoms of mass greater than α.)

THEOREM 1.2. *If $\vec{\mu} \in \mathcal{P}_n(\alpha)$, then $D(R(\vec{\mu})) \leq \alpha n/2$.*

THEOREM 1.3 (LYAPOUNOV [8]). *If $\mu_1, \ldots, \mu_n$ are nonatomic, then $R(\vec{\mu})$ is convex.*

The proof of Theorem 1.2 will be given in §3; Theorem 1.3 follows from Theorems 1.1 and 1.2 and the following easy lemma.

LEMMA 1.4. *A closed set $B \subset \mathbf{R}^n$ is convex if and only if $D(B) = 0$.*

2. Purely atomic measures. The purpose of this section is to prove some preliminary results corresponding to the case where each μ_i is purely atomic with only a finite number of atoms. Throughout this section, V is a finite set of (not necessarily distinct) points in $\mathbf{R}^n_+ = \{(r_1, \ldots, r_n) : r_i \in \mathbf{R}, r_i \geq 0$ for all $i \leq n\}$.

DEFINITION 2.1.

$$\Sigma(V) = \left\{ \sum_{x_i \in V} \delta_i x_i : \delta_i = 0 \text{ or } 1 \right\}; \qquad C(V) = \left\{ \sum_{x_i \in V} t_i x_i : t_i \in [0,1] \right\}.$$

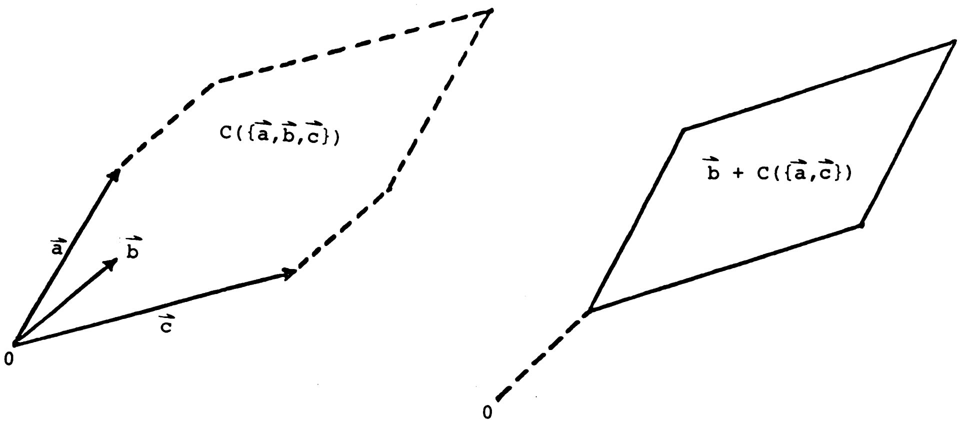

FIGURE 2

(So $\Sigma(V)$ is a finite set of points in $\mathbf{R}_+^n$ and $C(V)$ contains $\Sigma(V)$; see Figure 2. $C(V)$ is a *zonotope* [**1**], or *zonohedron* [**2**], and most of the results of this section may be rephrased using that terminology.)

LEMMA 2.2. $\mathrm{co}(\Sigma(V)) = C(V)$.

PROOF. Routine. □

The next lemma states that $C(V)$ can be expressed as the union of translates of subsets of the form $C(\hat{V})$ (see Figure 2) where $|\hat{V}| \leq n$. Its proof is similar to an argument of Carathéodory (see [**11**, p. 35]).

LEMMA 2.3. $C(V) = \bigcup\{\Sigma(V\backslash\hat{V}) + C(\hat{V})\colon \hat{V} \subset V, |\hat{V}| \leq n\}$.

PROOF. Clearly $C(V) \supset \bigcup\{\Sigma(V\backslash\hat{V}) + C(\hat{V})\colon V \subset \hat{V}, |\hat{V}| \leq n\}$, so fix $x = \sum_{i=1}^m t_i x_i \in C(V)$, where $\{x_i\}_{i=1}^m \subset V$ and $\{t_i\} \subset [0,1]$. If $m \leq n$, the conclusion is trivial, so suppose $m > n$. It will be shown that there exist $\{\tilde{t}_i\}_{i=1}^m \subset [0,1]$ with $\tilde{t}_j = 0$ or 1 for some $j \leq n$ so that $x = \sum_{i=1}^m \tilde{t}_i x_i$, and the conclusion will then follow by induction.

Assume further that $0 < t_i < 1$ for all $i \leq m$ (for otherwise taking $t_i = \tilde{t}_i$ suffices). Since $m > n$, there exist constants $\{a_i\}_{i=1}^m$ *not all zero*, so $\sum_{i=1}^m a_i x_i = 0$. Let

$$b = \min\{t_i/|a_i|, (1-t_i)/|a_i|\colon i = 1, \ldots, m,\ a_i \neq 0\},$$

and observe that $0 < b < \infty$.

CASE 1. $b = t_k/|a_k|$ for some $1 \leq k \leq m$. Let

$$\tilde{t}_i = t_i - (b \operatorname{sgn} a_k)a_i, \qquad i = 1, \ldots, m.$$

Note that $\tilde{t}_k = 0$, $\sum_{i=1}^m \tilde{t}_i x_i = \sum_{i=1}^m t_i x_i$, and if $a_i \neq 0$,

$$\tilde{t}_i \geq t_i - \frac{t_i}{|a_i|}|a_i| = 0 \quad \text{and} \quad \tilde{t}_i \leq t_i + \frac{(1-t_i)}{|a_i|}|a_i| = 1,$$

so $\tilde{t}_i \in [0,1]$ for each i (since if $a_i = 0$, $t_i = \tilde{t}_i$).

CASE 2. $b = (1-t_k)/|a_k|$ for some $1 \leq k \leq m$. Let

$$\tilde{t}_i = t_i + (b \operatorname{sgn} a_k)a_i, \qquad i = 1, \ldots, m.$$

Note $\tilde{t}_k = 1$, $\sum_{i=1}^m \tilde{t}_i x_i = \sum_{i=1}^m t_i x_i$, and check as before that $\{\tilde{t}_i\} \subset [0,1]$. □

LEMMA 2.4. *If $V \subset [0,1]^n$ and $|V| = n$, then the distance from any point in the parallelepiped $C(V)$ to the nearest vertex is $\leq n/2$.*

PROOF. First it will be shown that, for all $x, y \in \mathbf{R}^n$ and all $t \in [0,1]$,

$$\min\{\|x+(1-t)y\|^2, \|x-ty\|^2\} \leq \|x\|^2 + \|y\|^2/4. \tag{1}$$

To see (1), first consider the case $2\langle x,y\rangle \geq (t^2-(1-t)^2)\|y\|^2$. Then

$$\begin{aligned}\|x-ty\|^2 &= \|x\|^2 + t^2\|y\|^2 - 2t\langle x,y\rangle \\ &\leq \|x\|^2 + t^2\|y\|^2 - t(t^2-(1-t)^2)\|y\|^2 \\ &= \|x\|^2 + t(1-t)\|y\|^2 \leq \|x\|^2 + \|y\|^2/4.\end{aligned}$$

The case where $2\langle x,y\rangle \leq (t^2-(1-t)^2)\|y\|^2$ is similar, yielding $\|x+(1-t)y\|^2 \leq \|x\|^2 + \|y\|^2/4$.

Next, let $V = \{x_1, \ldots, x_n\}$ and fix $x = \sum_{i=1}^n t_i x_i \in C(V)$. Applying (1) n times implies the existence of $\{\delta_i\}_{i=1}^n \in \{0,1\}$ satisfying

$$\left\|\sum_{i=1}^n \delta_i x_i - x\right\|^2 = \left\|\sum_{i=1}^n (\delta_i - t_i)x_i\right\|^2 \leq n \cdot n/4 = n^2/4,$$

which completes the proof. □

PROPOSITION 2.5. *If $V \subset [0,\alpha]^n$, then $D(\Sigma(V)) \leq \alpha n/2$.*

PROOF. By Lemma 2.2 and the definition of D, $D(\Sigma(V)) = \sup\{d(x,\Sigma(V)) : x \in C(V)\}$, which together with Lemma 2.3 implies $D(\Sigma(V)) \leq \max\{D(\Sigma(\hat{V})) : \hat{V} \subset V, |\hat{V}| \leq n\}$. Then Lemma 2.4 (and rescaling) implies $D(\Sigma(\hat{V})) \leq \alpha n/2$ for all such $\hat{V}$. □

The next example shows that the bound in Lemma 2.4 (and hence in Proposition 2.5 and Theorem 1.2) is of the correct order in n; in fact the best possible bound (which is not known to the authors) is at least $n/8$ for general n and at least $n/4$ if n is a power of 2.

EXAMPLE 2.6. Fix n, let $m = 2^k \leq n < 2^{k+1}$, and let $\{w_i\}_{i=1}^{m-1}$ be the $m-1$ mean-zero *Walsh functions* on m points (see [**12**]). Then $w_i \in \{-1,1\}^m$, $w_i \perp w_j$ for $i \neq j$, and $w_i \perp \vec{1}$ for each i, where $\vec{1} = (1,1,\ldots,1)$. For example, when $n = 4$ (so $k = 2$ and $m = 4$)

$$w_1 = (1,1,-1,-1), \quad w_2 = (1,-1,1,-1), \quad \text{and} \quad w_3 = (1,-1,-1,1).$$

Let $x_i = (w_i + \vec{1})/2$ for $i = 1, \ldots, m-1$, so $x_i \in \{0,1\}^m \subset [0,1]^m$, and

$$\begin{aligned}\langle x_i, x_j\rangle &= \langle w_i + \vec{1}, w_j + \vec{1}\rangle/4 \\ &= (\langle w_i, w_j\rangle + \langle w_i, \vec{1}\rangle + \langle \vec{1}, w_j\rangle + \langle \vec{1}, \vec{1}\rangle)/4 \\ &= \begin{cases} m/4 & \text{if } i \neq j, \\ m/2 & \text{if } i = j.\end{cases}\end{aligned}$$

Let $V = \{x_1, \ldots, x_{m-1}\}$; it will now be shown that the distance from the center of $C(V)$ to the nearest vertex is at least $m/4$.

Let $\delta_i = 0$ or 1 for $i = 1, \ldots, m$, and let $\varepsilon_i = 1 - 2\delta_i$, so $\varepsilon_i = \pm 1$ for each i. Then

$$\begin{aligned} d^2\left(\sum_{i=1}^{m-1} x_i/2, \sum_{i=1}^{m-1} \delta_i x_i\right) &= \frac{1}{4}\left\|\sum_{i=1}^{m-1} \varepsilon_i x_i\right\|^2 \\ &= \frac{1}{4}\left(\sum_{i=1}^{m-1} \|x_i\|^2 + \sum\sum_{i\neq j} \varepsilon_i\varepsilon_j\langle x_i, x_j\rangle\right) \\ &= \frac{1}{4}\left(\frac{m(m-1)}{2} + \frac{m}{4}\sum\sum_{i\neq j}\varepsilon_i\varepsilon_j\right). \end{aligned}$$

Since $m-1$ is odd,

$$0 \leq \left(\sum_{i=1}^{m-1}\varepsilon_i\right)\left(\sum_{j=1}^{m-1}\varepsilon_j\right) - 1 = \sum\sum_{i\neq j}\varepsilon_i\varepsilon_j + (m-2),$$

so $\sum\sum_{i\neq j}\varepsilon_i\varepsilon_j \geq -(m-2)$. Thus

$$\frac{1}{4}\left\|\sum_{i=1}^{m-1}\varepsilon_i x_i\right\|^2 \geq \frac{1}{4}\left(\frac{m(m-1)}{2} - \frac{(m(m-2))}{4}\right) = \left(\frac{m}{4}\right)^2,$$

so $d(\sum_{i=1}^{m-1} x_i/2, \sum_{i=1}^{m-1}\delta_i x_i) \geq m/4$.

Since $m \leq n$, one may consider $x_i \in [0,1]^n$ for each i, and since $2m = 2^{k+1} > n$, the best possible upper bound in Lemma 2.4 is greater than $n/8$.

3. Proof of Theorem 1.2.

LEMMA 3.1. *For each $\vec{\mu}$ and each $\varepsilon > 0$, there exists a measurable partition $\{B_i\}_{i=1}^N$ of X satisfying*

(2) *for each $B \in \mathcal{F}$, $\exists J \subset \{1, \ldots, N\} \ni \left\|\vec{\mu}(B) - \vec{\mu}\left(\bigcup_{j\in J} B_j\right)\right\| < \varepsilon$.*

PROOF. Since $R(\vec{\mu})$ is bounded, there is an ε-net $\{x_1, \ldots, x_m\}$ of $R(\vec{\mu})$; that is, $\{x_1, \ldots, x_m\} \subset R(\vec{\mu})$ and for each $x \in R(\vec{\mu})$ there is an $i \leq m$ with $\|x - x_i\| < \varepsilon$. Let $\{A_i\}_{i=1}^m \in \mathcal{F}$ satisfy $\vec{\mu}(A_i) = x_i$, $i = 1, \ldots, m$, and let $\{B_i\}_{i=1}^N \in \mathcal{F}$ be disjoint with $\sigma(B_1, \ldots, B_N) = \sigma(A_1, \ldots, A_m)$. It is easily seen that $\{B_i\}_{i=1}^N$ satisfies (2). $\square$

LEMMA 3.2. *If $\vec{\mu} \in \mathcal{P}_n(\alpha)$, then for each $B \in \mathcal{F}$, $\exists$ a measurable partition $\{B_i\}_{i=1}^k$ of B such that $\mu_j(B_i) \leq \alpha$ for each $j \leq n$ and $i \leq k$.*

PROOF. Assume $\mu_1(B) > \alpha$. Let

$$\lambda = \inf\{\mu_1(E)\colon E \subset B,\ E \in \mathcal{F}, \text{ and } \mu_1(E) \geq \alpha/2\}.$$

By Theorem 1.1, λ is attained, that is, there exists $E_1 \in \mathcal{F}$ with $E_1 \subset B$ and $\mu_1(E_1) = \lambda \geq \alpha/2$. Next it will be shown that

$$\lambda \leq \alpha. \tag{3}$$

If $\lambda > \alpha$, then E_1 is not an atom of μ_1 (since $\mu_1 \in \mathcal{P}_1(\alpha)$), so there is a measurable subset C of E_1 with $0 < \mu_1(C) < \mu_1(E_1)$. But then either $\mu_1(C) > \alpha/2$

or $\mu_1(E_1 \backslash C) > \alpha/2$ since $\mu_1(E_1) > \alpha$. This contradicts the definition of λ, since $\mu_1(C) < \lambda$ and also $\mu_1(E_1 \backslash C) < \lambda$.

Repeat this argument for $B \backslash E_1$ in place of B, etc., and then for $\mu_2, \mu_3, \ldots, \mu_n$ to arrive at the desired partition $\{B_i\}_{i=1}^k$. $\square$

REMARK. Notice Theorem 1.1 is used here only in the case $n = 1$.

PROOF OF THEOREM 1.2. Fix $\vec{\mu} \in \mathcal{P}_n(\alpha)$. If $\vec{\mu}$ is purely atomic with only a finite number of atoms, then $R(\vec{\mu}) = \Sigma(V)$, where $V = \{\vec{\mu}(A) \colon A \text{ is an atom of } \vec{\mu}\} \subset [0, \alpha]^n$, and the conclusion follows by Proposition 2.5. For general μ, Lemmas 3.1 and 3.2 will be used to approximate this finite-atom case.

First assume $\alpha > 0$. It must be shown that

$$d(x, R(\vec{\mu})) \leq \alpha n/2 \quad \text{for all } x \in \operatorname{co}(R(\vec{\mu})). \tag{4}$$

Fix $\varepsilon > 0$. By Lemma 3.1 and repeated application of Lemma 3.2, there is a measurable partition $\{B_i\}_{i=1}^N$ of X satisfying both (2) and $\mu_j(B_i) \leq \alpha$ for all $j \leq n$ and $i \leq N$. Let $\vec{\mu}_0$ be the restriction of $\vec{\mu}$ to $\sigma(B_1, \ldots, B_N)$; then $\vec{\mu}_0$ is purely atomic with only a finite number of atoms, and $\vec{\mu}_0 \in \mathcal{P}_n(\alpha)$, so by Proposition 2.5

$$D(R(\vec{\mu}_0)) = D\left(\Sigma\left(\bigcup_{i=1}^N \vec{\mu}(B_i)\right)\right) \leq \frac{\alpha n}{2}. \tag{5}$$

Fix $x \in \operatorname{co}(R(\vec{\mu}))$; then $x = \sum_{i=1}^m t_i x_i$ for some $\{x_i\} \subset R(\vec{\mu})$ and some $\{t_i\} \geq 0$, $\sum_{i=1}^m t_i = 1$. By (2) there exist $J_1, \ldots, J_m \subset \{1, \ldots, N\}$ satisfying

$$\left\| x_i - \vec{\mu}\left(\bigcup_{j \in J_i} B_j\right) \right\| < \varepsilon \quad \text{for all } i = 1, \ldots, m, \tag{6}$$

so letting $y = \sum_{i=1}^m t_i \vec{\mu}\left(\bigcup_{j \in J_i} B_j\right) \in \operatorname{co}(R(\vec{\mu}_0))$,

$$\|x - y\| = \left\| \sum_{i=1}^m t_i \left(x_i - \vec{\mu}\left(\bigcup_{j \in J_i} B_j\right)\right)\right\| \leq \sum_{i=1}^m t_i \varepsilon = \varepsilon. \tag{7}$$

Next observe that

$$\begin{aligned} d(x, R(\vec{\mu})) &\leq d(x, y) + d(y, R(\vec{\mu})) \\ &\leq \varepsilon + d(y, R(\vec{\mu}_0)) \leq \varepsilon + \alpha n/2, \end{aligned} \tag{8}$$

where the second inequality in (8) follows from (7) and the fact that $R(\vec{\mu}_0) \subset R(\vec{\mu})$, and the third inequalilty from (5) since $y \in \operatorname{co}(R(\vec{\mu}_0))$. Since ε was arbitrary, this completes the proof of (4); the case $\alpha = 0$ follows easily by continuity. $\square$

4. Applications. Lyapounov's Convexity Theorem has been applied to a variety of problems in such diverse areas as Banach space theory, optimal stopping theory, control theory, and statistical decision theory (see Diestel and Uhl [**3**]); the purpose of this section is to mention two similar applications of Theorem 1.2.

BISECTION PROBLEM. If $\mu_1, \ldots, \mu_n$ are nonatomic probability measures on the same measurable space $(X, \mathcal{F})$, then there is always a measurable subset A of X with $\mu_i(A) = 1/2$ for all i (this is a special case of a theorem of Neyman [**9**]). The following theorem generalizes this result.

THEOREM 4.1. *If $\mu_1, \ldots, \mu_n$ are probability measures and $\mu_i \in \mathcal{P}_1(\alpha)$ for all i, then there is a measurable subset A of X satisfying*

$$\left|\mu_i(A) - \tfrac{1}{2}\right| \leq \alpha n/2 \quad \text{for all } i = 1, 2, \ldots, n.$$

PROOF. Since $(0, 0, \ldots, 0)$ and $(1, 1, \ldots, 1) \in R(\vec{\mu})$, $(\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}) \in \operatorname{co}(R(\vec{\mu}))$. Apply Theorem 1.2. □

BANG-BANG PRINCIPLE. Lyapounov's Convexity Theorem was used by LaSalle [**6**] to establish a principle in control theory which says that if an admissible steering function (in an absolutely continuous problem) can bring the system from one state to another in time t, then there is a "bang-bang" steering function that can do the same thing in the same time. A generalization of a particular form of this principle is given by the next theorem, which essentially says that in a system with point masses (or discontinuities or jumps in the process), given an arbitrary steering function there is always a bang-bang steering function which will bring the system within distance αn of the state arrived at by the given steering. (In the following theorem, the set M is viewed as the collection of all admissible steering functions, and M^0 as the set of bang-bang steering functions—see LaSalle [**6**].)

THEOREM 4.2. *Let $\vec{\mu}$ be a finite n-dimensional vector-valued Borel measure on $[0,1]$ with $\mu_i(\{t\}) \leq \alpha$ for all $i \leq n$ and all $t \in [0,1]$. Let M be the set of all real-valued Borel measurable functions f on $[0,1]$ satisfying $|f(t)| \leq 1$ for all $t \in [0,1]$, and let M^0 be the subset of M with $|f(t)| = 1$ for all $t \in [0,1]$. Define*

$$K = \left\{ \int f \, d\vec{\mu} \colon f \in M \right\}$$

and

$$K^0 = \left\{ \int f \, d\vec{\mu} \colon f \in M^0 \right\}.$$

Then K^0 is compact, $\operatorname{co}(K^0) = K$ and $D(K^0) \leq \alpha n$.

PROOF. By Theorems 1.1 and 1.2, $R(\vec{\mu})$ is compact and $D(R(\vec{\mu})) \leq \alpha n/2$. Letting $f_E(t) = 2I_E(t) - 1$ for each subset E of $[0,1]$, it is clear that $K^0 = 2R(\vec{\mu}) - \vec{\mu}([0,1])$, so K^0 is compact and $D(K^0) \leq \alpha n$.

It is clear that K and K^0 are unaffected if we consider (via the usual equivalence classes of $|\vec{\mu}|$-a.e. equal functions) M and M^0 to be subsets of $L_\infty(|\vec{\mu}|)$, where $|\vec{\mu}|$ is the total variation of $\vec{\mu}$. By the Krein-Milman theorem M is the weak*-closure of the convex hull of M^0, and it follows easily that, for $z = \int f \, d\vec{\mu} \in K$, z is in the closure of $\operatorname{co}(K^0)$. But K^0 is compact, so (by [**10**, Theorem 3.25]) $\operatorname{co}(K^0)$ is also compact, which implies $z \in \operatorname{co}(K^0)$. The opposite inclusion $\operatorname{co}(K^0) \subset K$ is trivial. □

REFERENCES

1. E. D. Bolker, *The zonoid problem*, Amer. Math. Monthly **78** (1971), 529–531.
2. H. Coxeter, *Regular polytopes* (2nd ed.), Macmillan, New York, 1963.
3. J. Diestel and J. J. Uhl, Jr., *Vector measures*, Math. Surveys, No. 15, Amer. Math. Soc., Providence, R.I., 1977.
4. P. R. Halmos, *The range of a vector measure*, Bull. Amer. Math. Soc. **54** (1948), 416–421.
5. ——, *Measure theory*, Van Nostrand, Princeton, N.J., 1950.

6. J. P. LaSalle, *The time optimal control problem*, Contributions to the Theory of Nonlinear Oscillations, Vol. V, Princeton Univ. Press, Princeton, N.J., 1960, pp. 1–24.
7. J. Lindenstrauss, *A short proof of Lyapounov's convexity theorem*, J. Math. Mech. **15** (1966), 971–972.
8. A. Lyapounov, *Sur les fonctions-vecteurs complétement additives*, Bull. Acad. Sci. URSS **6** (1940), 465–478.
9. J. Neyman, *Un théorème d'existence*, C. R. Acad. Sci. Paris **222** (1946), 843–845.
10. W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1973.
11. J. Stoer and C. Witzgall, *Convexity and optimization in finite dimensions*. I, Springer-Verlag, New York, 1970.
12. J. L. Walsh, *A closed set of normal orthogonal functions*, Amer. J. Math. **55** (1923), 5–24.

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