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RADON'S PROBLEM FOR SOME SURFACES IN R^n

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ABSTRACT. Radon's problem for a family of curves in R^2 has been generalized to a family of $(n-1)$-dimensional surfaces in R^n. The problem is posed as a set of integral equations. Solutions to these equations are given for paraboloids and cardioids, and for these cases the null spaces and consistency conditions have been found.

In R^n let $x = (x_1, x_2, \ldots, x_n)$ be a vector, let ξ, η be unit vectors, and let $\cdot$ denote the scalar product. Let $r = |x|$, $x = r\xi$, and let p be a nonnegative real number. For a fixed p, η the expression

$$r^\alpha \cos\{\alpha \cos^{-1}(\xi \cdot \eta)\} = p^\alpha, \qquad \alpha > 0, \tag{1}$$

represents an $(n-1)$-dimensional surface which is symmetrical about η and for which $r = p$ when $\xi = \eta$. Radon's problem is to determine a function $f(x)$ given the integrals of f over the surfaces (1). This is a generalization of Radon's problem in which (1) represented a family of curves in R^2, which was discussed in [**1**, **2**, **3**]. In this two-dimensional problem α was assumed to be positive and the curves were called α-curves, and for α negative we defined $\beta = -\alpha$ and referred to these curves as β-curves. The α-curves and β-curves are intimately related since an α-curve becomes the corrsponding β-curve under inversion in the unit circle: $(r, \theta) \to (1/r, \theta)$. In what follows we shall first consider the surfaces (1) for α as stated, namely, $\alpha > 0$, and then in Appendix A list the important formulas for the β-surfaces with corresponding formula numbers since the treatments are so similar that the arguments need not be repeated.

We first establish the integral equations for f when f is expanded in spherical harmonics. For $\alpha = 1/2$ these equations reduce to an extension of an integral equation which was solved by Wimp [**13**], and which is discussed in Appendix B. For $\alpha = 1/2$, (1) represents a family of paraboloids. Their closest point to the origin is $x = p\eta$, and for $p = 0$ they degenerate to the straight line from the origin to infinity in the direction $-\eta$. For $\alpha = -1/2$ ($\beta = 1/2$), (1) represents a family of cardioids. Their greatest distance from the origin is $x = p\eta$ and $r \to 0$ as $\xi \to -\eta$. The solution in this case depends on a further modification of Wimp's result, also discussed in Appendix B.

In addition to the solutions of Radon's problem for these paraboloids and cardioids we give the null-spaces and consistency conditions for the corresponding Radon transforms.

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We restrict ourselves to $n \geq 3$ because the results for $n = 2$ are known [**1**, **2**, **3**]. For many but perhaps not all of the results given below the formulas for $n = 2$ may be obtained by using the formulas

$$\lim_{\lambda \to 0} \Gamma(\lambda) C_l^\lambda(\cos\theta) = (2/l) T_l(\cos\theta) = (2/l)\cos l\theta,$$

where C_l^λ is a Gegenbauer polynomial and $T_l(x)$ is a Tchebycheff polynomial of the first kind [**9**].

Let $f(x)$ be a smooth, rapidly decreasing function of x andd let dx be an element of volume of R^n. The integral of f over the surface (1) for a particular (p, η) will be denoted by $\hat{f}(p, \eta)$ which will be called the Radon transform of f. The case $\alpha = 1$ is the well-known case of integrals over planes discussed by Ludwig [**8**] and Deans [**5**], who gives numerous other references.

Let $\delta(\cdot)$ be the Dirac delta-function and let $h(\xi \cdot \eta) = \cos\{\cos^{-1}(\xi \cdot \eta)\}$. Then

$$\alpha r^{\alpha - 1} \delta(r^\alpha h(\xi \cdot \eta) - p^\alpha) \tag{2}$$

represents a delta-function of unit weight concentrated on the surface (1). The factor $\alpha r^{\alpha-1}$ ensures that the expression has unit weight (Papoulis [**10**]), that is to say the integral

$$\hat{f}(p, \eta) = \int f(x) \alpha r^{\alpha-1} \delta(r^\alpha h(\xi \cdot \eta) - p^\alpha)\, dx \tag{3}$$

is the integral over this surface in standard measure, the integral being taken over all space.

Suppose that f is now expanded in spherical harmonics [**6**]:

$$f(x) = \sum_l f_{lm}(r) S_{lm}(\xi). \tag{4}$$

Then $f_{lm}(r) = r^l \times$ (an even function of r), and the Radon transform of a typical term in (4) may be denoted by $\hat{f}_{lm}(p, \eta)$ and written

$$\hat{f}_{lm}(p, \eta) = \int f_{lm}(r) S_{lm}(\xi) \alpha r^{\alpha-1} \delta(r^\alpha h(\xi \cdot \eta) - p^\alpha)\, dx. \tag{5}$$

In R^n, $dx = r^{n-1}\, dr\, d\Omega_\xi = r^{2\lambda+1}\, dr\, d\Omega_\xi$, where $\lambda = (n-2)/2$ and $d\Omega_\xi$ is an element of solid angle. Because $n \geq 3$, $\lambda \geq 1/2$. Thus (5) can be written

$$\hat{f}_{lm}(p, \eta) = \alpha \int_{\Omega_\xi} S_{lm}(\xi)\, d\Omega_\xi \int_0^\infty f_{lm}(r) r^{2\lambda+\alpha} \delta(r^\alpha h - p^\alpha)\, dr. \tag{6}$$

The r-integral is found to be $p^{2\lambda+1} f_{lm}(p/h^{1/\alpha})/\alpha h^{(2\lambda+\alpha+1)/\alpha}$ so (6) becomes

$$\hat{f}_{lm}(p, \eta) = p^{2\lambda+1} \int_{\Omega_\xi} f_{lm}(p/h^{1/\alpha}) S_{lm}(\xi) h^{-(2\lambda+\alpha+1)/\alpha}\, d\Omega_\xi. \tag{7}$$

Since $h = h(\xi \cdot \eta)$, the integral is of the form $\int_{\Omega_\xi} G(\xi \cdot \eta) S_{lm}(\xi)\, d\Omega_\xi$ which, according to the Funk-Hecke theorem [**6**], is equal to $\beta_l S_{lm}(\eta)$ where

$$\beta_l = \frac{\omega_{n-1}}{C_l^\lambda(1)} \int_{-1}^{+1} G(t) C_l^\lambda(t)(1 - t^2)^{\lambda - 1/2}\, dt, \tag{8}$$

in which ω_{n-1} is the area of the unit sphere in R^{n-1} and $C_l^\lambda(t)$ are the Gegenbauer or ultraspherical polynomials [**6**, **9**]. Since β_l is a function of p only we may write

$$\hat{f}_{lm}(p,\eta) = \hat{f}_l(p) S_{lm}(\eta), \tag{9}$$

where we hve suppressed the m in $\hat{f}_l(p)$, and where

$$\hat{f}_l(p) = p^{2\lambda+1}\frac{\omega_{n-1}}{C_l^\lambda(1)}\int_{\cos(\pi/2\alpha)} f_l\left(\frac{p}{h^{1/\alpha}}\right) \times [h(t)]^{-(2\lambda+\alpha+1)/\alpha} C_l^\lambda(t)(1-t^2)^{\lambda-1/2}dt, \tag{10}$$

and the limits result from the specific form of $h(t)$, namely, $h(t) = \cos\{\alpha\cos^{-1} t\}$.

On changing the integration variable from t to $r = p/h^{1/\alpha}$, and on further putting $r^\alpha = s$ and $p^\alpha = q$ and defining

$$F_l(s) = (1/\alpha) f_l(s^{1/\alpha})s^{1/\alpha-1}, \qquad \hat{F}_l(q) = \hat{f}_l(q^{1/\alpha}), \tag{11}$$

(10) becomes

$$\hat{F}_l(q) = \frac{\omega_{n-1}}{C_l^\lambda(1)}\int_q^\infty F_l(s)s^{(2\lambda/\alpha)}C_l^\lambda\left(\cos\left\{\frac{1}{\alpha}\cos^{-1}\left(\frac{q}{s}\right)\right\}\right) \times \sin^{2\lambda}\left\{\frac{1}{\alpha}\cos^{-1}\left(\frac{q}{s}\right)\right\}\left(1-\left(\frac{q}{s}\right)^2\right)^{-1/2} ds. \tag{12}$$

The solution of this integral equation for $F_l(s)$ will solve Radon's problem for these surfaces. Quinto [**11**] has remarked that (12) is invertible for functions of compact support and that this implies a hole theorem (see below). No inversion formula has been found except for the special cases $\alpha = 1/2, -1/2$ (paraboloids, cardioids) for which (12) can be reduced to a hypergeometric integral equation with a known solution. We proceed to discuss this solution and to give some properties of the $\hat{f}_l$. For $\alpha = 1/2$, (12) becomes

$$\hat{F}_l(q) = \frac{2\omega_{n-1}}{C_l^\lambda(1)}\int_q^\infty f_l(s^2)s^{4\lambda}C_l^\lambda\left(\cos\left\{2\cos^{-1}\left(\frac{q}{s}\right)\right\}\right) \times \sin^{2\lambda}\left\{2\cos^{-1}\left(\frac{q}{s}\right)\right\}\left(1-\left(\frac{q}{s}\right)^2\right)^{-1/2} s\, ds. \tag{13}$$

Let $F(\cdot,\cdot;\cdot;\cdot)$ denote the ordinary hypergeometric function ${}_2F_1$. Then it is known [**9**] that

$$C_l^\lambda(x) = \frac{\Gamma(l+2\lambda)}{l!\Gamma(2\lambda)}F\left(-l, l+2\lambda; \lambda+\frac{1}{2}; \frac{1-x}{2}\right). \tag{14}$$

If $\theta = \cos^{-1}(q/s)$, then use of the formulas for $\cos 2\theta$ and $\sin 2\theta$ enables (13) to be written

(15)

$$\hat{F}_l(q) = \frac{(2q)^{2\lambda}\Gamma(l+2\lambda)2\omega_{n-1}}{l!\Gamma(2\lambda)C_l^\lambda(1)}\int_q^\infty f_l(s^2)s^{2\lambda}F\left(-l, l+2\lambda; \lambda+\frac{1}{2}; 1-\left(\frac{q}{s}\right)^2\right) \times \left(1-\left(\frac{q}{s}\right)^2\right)^{\lambda-1/2} s\, ds,$$

and the further substitutions $s^2 = t$, $q^2 = r$ reduce (15) to

$$\frac{\hat{f}_l(r)}{r^\lambda} = K_l \int_r^\infty f_l(t) t^\lambda F\left(-l, l+2\lambda; \lambda + \frac{1}{2}; 1 - \frac{r}{t}\right)\left(1 - \frac{r}{t}\right)^{\lambda - 1/2} dt, \tag{16}$$

where

$$K_l = \frac{2^{2\lambda}\Gamma(l+2\lambda)\omega_{n-1}}{l!\Gamma(2\lambda)C_l^\lambda(1)}. \tag{17}$$

For $l = 0$ the hypergeometric function is 1 and the resulting integral equation is easily solved by differentiation if $\lambda = 1/2, 3/2, \ldots$, or is reduced to Abel's equation by differentiation when $\lambda = 1, 2, 3, \ldots$. In what follows $l > 0$.

If we make the identification $a = -l$, $b = l + 2\lambda$, and $c = \lambda + 1/2$, then (16) is precisely of the form of the modified Wimp equation (B3). The solution to (B3) is (B7), which contains an integer $m > c > 0$. In the case of planes [**5**, **8**] and spheres through the origin [**4**] the derivative of $\hat{f}_l$ which appears in the solutions to the analog of (16) is the $(n-1)$th derivative, so we choose $m = n - 1 = 2\lambda + 1$. The solution of (16) is then

(18)
$$\frac{f_l(x)}{\sqrt{x}} = \frac{(-1)^{2\lambda+1}}{(4\pi)^{\lambda+1/2}\Gamma(\lambda+1/2)} \int_x^\infty (y-x)^{\lambda-1/2} F\left(l, -(l+2\lambda); \lambda + \frac{1}{2}; 1 - \frac{y}{x}\right) \times \frac{d^{n-1}}{dy^{n-1}}\left(\frac{\hat{f}_l(y)}{y^\lambda}\right) dy.$$

(It may be noted that Wimp's result can be used directly for the α-curves in R^2 to avoid the clumsy derivation given in [**1**].) (18) illustrates the hole theorem: in order to find f at x_0 it is only necessary to know $\hat{f}$ for $|y| \geq x_0$.

We now obtain for this paraboloidal Radon transform results about its null space, and the so-called consistency conditions, both of which are extensions to R^n of results in R^2 [**1**, **2**]. We rewrite (16) using the result [**9**]

$$F\left(-l, l+2\lambda; \lambda+\frac{1}{2}; 1-x\right) = \frac{\Gamma(\lambda+1/2)\Gamma(2\lambda+2l)}{\Gamma(\lambda+l+\frac{1}{2})\Gamma(l+2\lambda)} G_l\left(2\lambda, \lambda+\frac{1}{2}, x\right), \tag{19}$$

where $G_l(p, q, x)$ is a shifted Jocobi polynomial. These polynomials form a complete set in $(0,1)$ with a weight function $w(x) = x^{q-1}(1-x)^{p-q}$, $q > 0$, $p - q > -1$. The weight function for $G_l(2\lambda, \lambda + 1/2, x)$ is thus

$$w_\lambda(x) = x^{\lambda-1/2}(1-x)^{\lambda-1/2}. \tag{20}$$

(16) becomes

(21)
$$\frac{\hat{f}_l(r)}{r^\lambda} = (4\pi)^\lambda 2^{2(l+\lambda)} \frac{\Gamma(l+\lambda)}{\Gamma(l+2\lambda)} \int_r^\infty f_l(t) t^\lambda G_l\left(2\lambda, \lambda+\frac{1}{2}, \frac{r}{t}\right)\left(1-\frac{r}{t}\right)^{\lambda-1/2} dt$$

which, with a change of variable, may be written

(22)
$$\frac{\hat{f}_l(r)}{r^{2\lambda+1}} = (4\pi)^\lambda 2^{2(l+\lambda)} \frac{\Gamma(l+\lambda)}{\Gamma(l+2\lambda)} \int_0^1 f_l\left(\frac{r}{x}\right) x^{-2\lambda-3/2} G_l\left(2\lambda, \lambda+\frac{1}{2}, x\right) w_\lambda(x)\, dx.$$

If $f(t) = t^{-k}$, $f_l(r/x)x^{-2\lambda-3/2} = x^{k-2\lambda-3/2}r^{-k}$, and because G_l is orthogonal to any polynomial in x of degree less than l, the integral in (22) will vanish for $k = l+2\lambda-1/2, l+2\lambda-3/2, \ldots, 2\lambda+3/2$. Thus the null space contains at least powers of t with these exponents.

To find the consistency conditions multiply (18) by r^μ and integrate:

$$\begin{aligned}(23)\quad \int_0^\infty \hat{f}_l(r)r^{\mu-\lambda}\,dr &= \text{const}\int_0^\infty r^\mu\,dr\int_r^\infty \hat{f}_l(t)t^\lambda G_l\left(2\lambda,\lambda+\frac{1}{2},\frac{r}{t}\right)^{\lambda-1/2}dt \\ &= \text{const}\int_0^\infty f_l(t)t^{\lambda+\mu+1}dt\int_0^1 x^{\mu-\lambda+1/2}G_l\left(2\lambda,\lambda+\frac{1}{2},x\right)w_\lambda(x)\,dx.\end{aligned}$$

Again using the orthogonality of the G_l, the x integral will vanish if $\mu-\lambda = l-3/2$, $l-5/2,\ldots,-1/2$. Thus the consistency conditions may be written

$$(24)\qquad \sum_{m=0}^{l-1} a_m\int_0^\infty \hat{f}_l(r)r^{l-(3/2)-m}\,dr = 0,$$

where the a_m are l arbitrary numbers, and (24) may be integrated by parts $(n-1)$ times to give

$$(25)\qquad \sum_{k=0}^{l-1}\alpha_k\int_0^\infty \hat{f}_l^{(n-1)}(r)r^{\lambda+(1/2)+k}\,dr,$$

where the α_k are other arbitrary constants.

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Appendix A. Equation numbers in this section such as (An) are the analogs of (n) in the main text.

The β-curves are defined by the equation

$$(\text{A1})\qquad r^\beta = p^\beta\cos\{\beta\cos^{-1}(\xi\cdot\eta)\} = p^\beta h(\xi\cdot\eta),$$

and the delta-function of unit weight which is concentrated on these surfaces in

$$(\text{A2})\qquad (\beta p^\beta/r)\delta(r^\beta - p^\beta h(\xi\cdot\eta)).$$

On expanding f and $\hat{f}$ in spherical harmonics, applying the Funk-Hecke theorem, and defining F_l and $\hat{F}_l$ as in (11) we obtain the integral equation for the β-surfaces:

$$\begin{aligned}(\text{A13})\quad \hat{F}_l(q) = \frac{\omega_{n-1}}{C_l^\lambda(1)}\int_0^q F_l(s)C_l^\lambda&\left(\cos\left\{\frac{1}{\beta}\cos^{-1}\left(\frac{s}{q}\right)\right\}\right) \\ &\times\sin^{2\lambda}\left\{\frac{1}{\beta}\cos^{-1}\left(\frac{s}{q}\right)\right\}s^{2\lambda/\beta}\left(1-\frac{s^2}{q^2}\right)^{-1/2}ds.\end{aligned}$$

For $\beta = 1/2$ (cardioids) (A13) reduces to

$$(\text{A16})\qquad r^\lambda\hat{f}_l(r) = K_l\int_0^r f_l(t)t^{3\lambda}F(-l,l+2\lambda;\lambda+1/2;1-t/r)(1-t/r)^{\lambda-1/2}\,dt.$$

This is precisely of the form of the integral equation (B8) given in Appendix B and we can write down its solution from (B9):

$$
\begin{aligned}
p^{5\lambda+1} f_l(p) = {} & \frac{(-1)^n}{(4\pi)^{\lambda+1/2}\Gamma(\lambda+1/2)} \\
& \times \int_0^p \frac{d^{n-1}}{dy^{n-1}}\{y^{3\lambda}\hat{f}_l(y)\} F\left(l, -l-2\lambda; \lambda+\frac{1}{2}; \left(1-\frac{p}{y}\right)\right) \\
& \times \left(\frac{p}{y}-1\right)^{\lambda-1/2} y^{2\lambda}\, dy,
\end{aligned} \tag{A18}
$$

where a, b, c, and m have been identified as they were for $\alpha = 1/2$. (A18) illustrates the "hole"-theorem for this case: in order to find f for some value of p it is only necessary to know $\hat{f}$ for $|y| \leq p$.

In (A16), $F(\cdot, \cdot; \cdot; \cdot)$ may again be expressed as a shifted Jacobi polynomial and use of the orthogonality of these polynomials yields the following information about the null-space of the transform and its consistency conditions:

(A22) the null-space of the transform contains $f_l(r) = r^\gamma$, where $\gamma = l - 2\lambda - 3/2, l - 2\lambda - 5/2, \ldots, -2\lambda - 1/2$,

$$\int_0^\infty \hat{f}_l(r) r^{-s}\, dr = 0 \quad \text{if } s = l - 1/2, l - 3/2, \ldots, 3/2. \tag{A23}$$

Appendix B. An extension of a result due to Wimp [**13**].

Sneddon [**12**, p. 295, Problem 4-21] gives an identity which may be rewritten

$$
\begin{aligned}
& \int_s^t (u-s)^{c-1}(t-u)^{m-c-1} F\left(a, b; c; 1-\frac{u}{s}\right) F\left(-a, -b; m-c; 1-\frac{u}{t}\right) u^{-m}\, du \\
& \qquad = \frac{\Gamma(c)\Gamma(m-c)}{\Gamma(m)} \frac{s^{c-m}(t-s)^{m-1}}{t^c},
\end{aligned} \tag{B1}
$$

where $0 < c < m =$ integer. The substitution $u = st/v$ in (B1) gives

$$
\begin{aligned}
& \int_s^t (t-v)^{c-1}(v-s)^{m-c-1} F\left(a, b; c; 1-\frac{t}{v}\right) F\left(-a, -b; m-c; 1-\frac{s}{v}\right) dv \\
& \qquad = \frac{\Gamma(c)\Gamma(m-c)}{\Gamma(m)} (t-s)^{m-1}.
\end{aligned} \tag{B2}
$$

Consider the integral equation

$$H(y) = \int_y^\infty (x-y)^{c-1} F(a, b; c; 1 - y/x) G(x)\, dx, \tag{B3}$$

where $G(x)$ is a smooth rapidly decreasing function of x. Multiply (B3) by $F(-a,-b;m-c;1-y/t)(y-c)^{m-c-1}y^{-m}$ and integrate y from t to ∞. Interchanging the order of integration on the R.H.S. yields

$$\begin{aligned}
&\int_t^\infty H(y)F(-a,-b;m-c;1-y/t)(y-c)^{m-c-1}y^{-m}\,dy \\
&\quad = (-1)^{m-1}\int_t^\infty G(x)\,dx \\
&\qquad \times \int_x^t F(-a,-b;m-c;1-y/t)F(a,b;c;1-y/x) \\
&\qquad\qquad \times (t-y)^{m-c-1}(y-x)^{c-1}y^{-m}\,dy.
\end{aligned} \tag{B4}$$

Use of (B1) gives

$$\begin{aligned}
&\frac{t^c\Gamma(m)}{\Gamma(c)\Gamma(m-c)}\int_t^\infty H(y)F\left(-a,-b;m-c;1-\frac{y}{t}\right)(y-t)^{m-c-1}y^{-m}\,dy \\
&\quad = \int_t^\infty x^{c-m}G(x)(x-t)^{m-1}\,dx = I,
\end{aligned} \tag{B5}$$

and since $d^mI/dt^m = (-1)^m(m-1)!G(t)t^{c-m}$, we have

$$\begin{aligned}
G(t)t^{c-m} = \frac{(-1)^m}{\Gamma(c)\Gamma(m-c)}\frac{d^m}{dt^m}\int_t^\infty H(y)F\left(-a,-b;m-c;1-\frac{y}{t}\right) \\
\times (y-t)^{m-c-1}t^c y^{-m}\,dy.
\end{aligned} \tag{B6}$$

The derivative may be taken under the integral sign to yield, as the solution of (B3),

(B7)

$$G(t) = \frac{(-1)^m}{\Gamma(c)\Gamma(m-c)}\int_t^\infty H^{(m)}(y)F\left(-a,-b;m-c;1-\frac{y}{t}\right)(y-t)^{m-c-1}\,dy,$$

where $H^{(m)}$ indicates the mth derivative of H.

Consider now the integral equation

$$H(y) = \int_0^y (y-x)^{c-1}F(a,b;c;1-x/y)G(x)\,dx. \tag{B8}$$

Using the same procedure and the identity (B2) we find the solution

$$\begin{aligned}
t^mG(t) = \frac{(-1)^{m-1}}{\Gamma(c)\Gamma(m-c)}\int_0^p \{y^{m-c}H(y)\}^{(m)}(p-y)^{m-c-1} \\
\times F\left(-a,-b;m-c;1-\frac{p}{y}\right)y^c\,dy.
\end{aligned}$$

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