PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 99, Number 2, February 1987

A MAXIMUM PRINCIPLE FOR QUOTIENT NORMS IN H^∞

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ABSTRACT. Let G be a closed subset of the open unit disk D in the complex plane, and let p denote a general polynomial of degree n which has all of its roots in G. For a fixed h in H^∞, $\|h - pH^\infty\|_{H^\infty/pH^\infty}$ is maximized only if all the zeros of p are on the boundary of G.

In studying a problem on the spectral radius of matrices, V. Pták was led to the following extremal problem: Let H^∞ denote the space of bounded analytic functions on the open unit disk D of the complex plane $\mathbf{C}$, and let h be a fixed function in H^∞. Among all polynomials p of degree n whose zeros are in $\{z \colon |z| \le r\}$ for a fixed $r < 1$, find one which maximizes $\|h - pH^\infty\|$ in the quotient space H^∞/pH^∞ (see [**2**, **5**]). Actually, Pták considered the case when h is of the form $h(z) = z^m$ for a fixed integer m. In fact, he showed that in the case $m = n$, the extremal polynomial can be taken to be $(z - r)^n$. It was conjectured that this is true for $m > n$ as well as that, for the general h in H^∞, each extremal p has all of its zeros on the circle $\{|z| = r\}$. The latter conjecture was recently proved by N. J. Young [**4**] in the special case that h is a Blaschke product of degree n, though the conjecture remained open even in the case $h = z^m$ for $m > n$. The contribution of this paper is to prove the following maximum principle for the extremal polynomial.

THEOREM 1. *Let G be a closed subset of D and let p denote a general polynomial of degree n which has all of its roots in G. For a fixed h in H^∞, let $F(p) = \|h - pH^\infty\|_{H^\infty/pH^\infty}$. If F is not constant as p varies, then it attains its maximum at p only if all the zeros of p lie on the boundary of G.*

The work of Pták and Young mentioned above has been largely operator-theoretic. In contrast, the present treatment is completely elementary, relying on the Schur algorithm for the solution of the Nevanlinna-Pick interpolation problem (see [**1**, **3**]). Since the treatment here is somewhat nonstandard, a brief description of the Schur algorithm is given below.

Let $a_1, a_2, \ldots, a_n$ be points in D and $W_1, W_2, \ldots, W_n$ the values to be interpolated along the a_j by a function f in the unit ball Σ of H^∞. We allow repetitions in the a_j's as long as they occur consecutively. For each k, let d_k denote the number of times $a_j = a_k$ for $j < k$. We are looking for a function f in Σ which satisfies

$$f^{(d_k)}(a_k) = w_k, \qquad k = 1, 2, \ldots, n.$$

The Schur algorithm proceeds inductively as follows. Suppose that f is in Σ and fits the given data. Take a_1. If $|w_1| > 1$, no solution exists. If $|w_1| \le 1$, the function

Received by the editors July 26, 1985 and, in revised form, January 10, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30C80, 30D55; Secondary 47A10.

f_1 defined by

$$f_1 = \frac{f - w_1}{z - a_1} \cdot \frac{1 - \bar{a}_1 z}{1 - \bar{w}_1 f} \tag{1}$$

belongs to Σ. Writing $b_1 = (z - a_1)/(1 - \bar{a}_1 z)$, we get that

$$f = \frac{b_1 f_1 + w}{1 + \bar{w}_1 b_1 f_1} \tag{2}$$

and no matter what f_1 in Σ is, the right hand expression above reduces to w_1 when evaluated at $z = a_1$. It is easily checked that, for any positive integer k, the first k derivatives of f at a_1 can be determined by the first $k-1$ derivatives of f_1 at a_1 and vice-versa. Likewise, at any other node, the interpolation data for f determine interpolation data for f_1 and vice-versa. So the problem of finding f in Σ reduces to solving a lower order interpolation for f_1 with revised data. If one proceeds inductively, there are three possibilities.

(i) The process reveals at some point that no solution exists in Σ.

(ii) The process terminates at the jth stage $(0 \leq j \leq n-1)$ yielding a unique solution. This solution is a Blaschke product of degree j. Conversely, if a Blaschke product of order $j \leq n-1$ is among the solutions, then the process terminates at the jth stage.

(iii) The process can be carried through the nth stage in which case the choice of f_n is indeterminate.

Now suppose the interpolation problem has a solution in Σ for the data $\{a_1, \ldots, a_{n-1}; w_1, \ldots, w_{n-1}\}$. Let a_n in D be added. Then the set of possible w_n for which the problem with data $\{a_1, \ldots, a_n; w_1, \ldots, w_n\}$ can be solved in Σ is a closed disk whose center and radius are determined by $a_1, a_2, \ldots, a_n$ and $w_1, w_2, \ldots, w_{n-1}$. The augmented interpolation problem has a unique solution if and only if w_n belongs to the boundary of that disk (see [**3**] or Chapter 1 of [**1**]). The disk reduces to a point if and only if condition (ii) is met on or before the $(n-2)$th stage.

We need to take a closer look at the Schur algorithm. Let

$$\Sigma(a_1, \ldots, a_n; w_1, \ldots, w_n) = \{f \in \Sigma\colon f^{(d_k)}(a_k) = w_k, k = 1, 2, \ldots, n\}$$

and

$$D(a_1, \ldots, a_{k+1}; w_1, \ldots, w_k) = \{f^{(d_{k+1})}(a_{k+1})\colon f \in \Sigma(a_1, \ldots, a_k; w_1, \ldots, w_k)\}.$$

Then, from (1) and (2),

$$\Sigma(a_1, \ldots, a_n; w_1, \ldots, w_n) = \left\{ f = \frac{b_1 \varphi + w_1}{1 + \bar{w}_1 b_1 \varphi} : \varphi \in \Sigma(a_2, \ldots, a_n; \hat{w}_2, \ldots, \hat{w}_n) \right\},$$

where the $\hat{w}_j$ are computed as follows:

Case 1. $(a_j \neq a_1)$. The first d derivatives of φ at a_j are determined by the first d derivatives of f and vice-versa. Thus,

$$\hat{w}_j = [(f - w_1)/(1 - \bar{w}_1 f) b_1]^{(d_j)}(a_j)$$

for any f in $\Sigma(a_1, \ldots, a_j;\ w_1, \ldots, w_j)$.

Case 2. $(a_j = a_1)$. From (1), letting $g = (1 - \bar{a}_1 z)/(1 - \bar{w}_1 f)$, we have

$$\hat{w}_j = \sum_{m=0}^{j-1} \binom{j-1}{m} f^{(m+1)}(a_1) g^{(j-1-m)}(a_1) = \sum_{m=0}^{j-1} \binom{j-1}{m} w_{m+1} g^{(j-1-m)}(a_1).$$

Hence, in either case, if $\Sigma(a_1, \dots, a_j; w_1, \dots, w_j)$ is nontrivial (i.e., contains more than one function), then $\hat{w}_2, \dots, \hat{w}_j$ vary continuously with $w_1, \dots, w_j$.

Though it is not used in the proof of Theorem 1, the following proposition is of interest in its own right.

PROPOSITION 1. *Suppose $D(a_1, \dots, a_{k+1}; w_1, \dots, w_k)$ has nonempty interior. Then it is a disk whose center and radius vary continuously with $w_1, w_2, \dots, w_k$.*

PROOF. (INDUCTION ON k). For $k = 1$, there are two cases:
If $a_2 \neq a_1$,

$$D(a_1, a_2; w_1) = \{[b_1(a_2)w + w_1]/[1 + \bar{w}_1 b_1(a_2)w] \colon |w| \leq 1\}.$$

If $a_2 = a_1$,

$$D(a_1, a_1; w_1) = \{w[1 - |w_1|^2]/[1 - |a_1|^2] \colon |w| \leq 1\}.$$

Now suppose that the lemma holds for $D_{j+1} \equiv D(a_1, \dots, a_{j+1};\ w_1, \dots, w_j)$ whenever $j < k$, and suppose that D_{k+1} has nonempty interior. Then

$$\begin{aligned} D_{k+1} &= \{f^{(d_{k+1})}(a_{k+1}) \colon f \in \Sigma(a_1, \dots, a_k; w_1, \dots, w_k)\} \\ &= \{[(b_1\varphi + w_1)/(1 + \bar{w}_1 b_1 \varphi)]^{(d_{k+1})}(a_{k+1}) \colon \\ &\qquad\qquad \varphi \in \Sigma(a_2, \dots, a_k; \hat{w}_2, \dots, \hat{w}_k)\}. \end{aligned} \tag{3}$$

If $d_{k+1} = 0$, then D_{k+1} is a Moebius transformation of $D(a_2, \dots, a_{k+1}; \hat{w}_2, \dots, \hat{w}_k)$ with parameter w_1 so that the desired result holds by the continuity of the $\hat{w}_j$ and the inductive hypothesis. For the case $1 \leq d_{k+1} < k$, let f be a solution to the kth order interpolation problem, and let φ be related to f as in (3). Then $[1 + \bar{w}_1 b_1 \varphi] f = b_1 \varphi + w_1$. Suppressing the subscript $k + 1$, we have

$$\begin{aligned} f^{(d)}(a)[1 + \bar{w}_1 b_1(a)\varphi(a)] = &- \sum_{m=0}^{d-1} \binom{d}{m} f^{(m)}(a)[1 + \bar{w}_1 b_1 \varphi]^{(d-m)}(a) \\ &+ \sum_{m=0}^{d} \binom{d}{m} b_1^{(d-m)}(a)\varphi^{(m)}(a), \\ f^{(d_{k+1})}&(a_{k+1}) = R + C\hat{w}, \end{aligned}$$

where $\hat{w} \in D(a_2, \dots, a_{k+1}; \hat{w}_2, \dots, \hat{w}_k)$ and where R and C are rational functions of $w_1, \dots, w_k$, $\hat{w}_2, \dots, \hat{w}_k$. By the inductive assumption and the continuity of the $\hat{w}_j$, the desired result again follows. The last remaining case $d_{k+1} = k$ is treated in a similar fashion.

The next two propositions are used in the proof of Theorem 1.

PROPOSITION 2. *Suppose that $\Sigma(a_1, \dots, a_n; w_1, \dots, w_n)$ contains a Blaschke product B of order $m \leq n - 1$. Then there exists $\delta_0 > 0$ such that whenever $|w_j - w'_j| < \delta < \delta_0$ for $j = 1, 2, \dots, m$, then $\Sigma(a_1, \dots, a_m;\ w'_1, \dots, w'_m)$ contains a Blaschke product b (which depends on the w'_j) of order m. Moreover, b can be chosen so that $\|B - b\|_\infty \to 0$ as $\sup\{|w_j - w'_j| \colon 1 \leq j \leq m\} \to 0$.*

PROOF. If $m = 1$, then

$$B = (b_1 \hat{w}_2 + w_1)/(1 + \bar{w}_1 b_1 \hat{w}_2)$$

so the desired conclusion follows from the continuity of $\hat{w}_2$. We now proceed inductively. If $\Sigma(a_1, \ldots, a_n;\, w_1, \ldots, w_n)$ contains a Blaschke product of order $m \leq n-1$, then $D(a_1, \ldots, a_{m+1};\, w_1, \ldots, w_m)$ is a nondegenerate disk and w_{m+1} belongs to its boundary. Also, $B = (b_1\hat{B} + w_1)/(1 + \bar{w}_1 b_1 \hat{B})$, where $\hat{B}$ is a Blaschke product of order $m-1$ in $\Sigma(a_2, \ldots, a_{m+1}; \hat{w}_2, \ldots, \hat{w}_{m+1})$. By induction, there exists $\eta_0 > 0$ such that whenever $|\hat{w}_j' - \hat{w}_j| < \eta < \eta_0$ for $j = 2, \ldots, m$, then there exists a Blaschke product $\hat{b}$ of order $m-1$ in $\Sigma(a_2, \ldots, a_m;\, \hat{w}_2', \ldots, \hat{w}_m')$ and such that $\|\hat{B} - \hat{b}\|_\infty \to 0$ as $\sup\{|\hat{w}_j - \hat{w}_j'| \colon j = 2, \ldots, m\} \to 0$. The desired result now follows from the continuity of the $\hat{w}_j$ as functions of $w_1, \ldots, w_j$.

Also needed will be the following well-known connection between interpolation and approximation theory. Suppose that b is a Blaschke product with zero sequence $a_1, \ldots, a_n$ (all repetitions are assumed to be consecutive), and let h be a function in H^∞. Then a function f in H^∞ is said to interpolate h along the zeros of b if $f - h$ belongs to bH^∞. Of course, this means that h is a solution to the interpolation problem with data $\{a_1, \ldots, a_n; w_1, \ldots, w_n\}$ where each w_k is a derivative of appropriate order of h at a_k. Now let $d = \operatorname{dist}(h, bH^\infty)$ which is defined by

$$\operatorname{dist}(h, bH^\infty) = \inf\{\|h - bg\|_\infty \colon g \text{ is in } H^\infty\}.$$

Then d is characterized in terms of interpolation of h along the zeros of b as follows.

PROPOSITION 3. *Let h be in H^∞ and let b be a Blaschke product of order n. A positive number c equals* $\operatorname{dist}(h, bH^\infty)$ *if and only if h/c can be interpolated along the zeros of b by a Blaschke product B of order at most $n-1$. Alternatively,* $\operatorname{dist}(h, bH^\infty)$ *can be characterized as the least real number $t \geq 0$ such that $h - tg$ belongs to bH^∞ for some g in Σ.*

PROOF. If $\operatorname{dist}(h, bH^\infty) = 0$ there is nothing to prove. Suppose that B is a Blaschke product of order $\leq n-1$, and that $h/c - B$ is in bH^∞. Assume that $d = \operatorname{dist}(h, bH^\infty) < c$. Then there is a function g in H^∞ such that $\|h - bg\|_\infty < |cB|$ on the boundary of D. Thus, $cB - (h - bg)$ is in bH^∞ and, by Rouché's theorem, has at most $n-1$ zeros in D counting multiplicity. This is absurd, so $c \leq d$. To establish the opposite inequality, note that $h - cB = bf$ for some f in H^∞. Thus, $c = \|h - bf\|_\infty \geq \operatorname{dist}(h, bH^\infty) = d$. The proof is concluded by the observation that for $c = \inf\{r > 0 \colon (h/r - bH^\infty) \cap \Sigma \neq \varnothing\}$, there exists a Blaschke product B of order $\leq n-1$ such that $h/c - B$ is in bH^∞ (this follows from the Schur algorithm).

Theorem 1 will now be proved by induction on the number of zeros of p. If $p(z) = z - a$, where a is in the interior of G, then $\operatorname{dist}(h, pH^\infty) = |h(a)|$ since $h - h(a) \in pH^\infty$ and, for any g in H^∞, $\|h - pg\|_\infty \geq |h(a)|$. If h is not constant, then there is an a' near a in G such $|h(a')| > |h(a)|$.

Assume now that the theorem has been established for all p with at most $n-1$ zeros. Let p have zero sequence $a_1, \ldots, a_n$ in G listed according to our convention on repetitions, and assume that a_n is in the interior of G. We shall show that either $h = cB$, where c is a constant and where B is a Blaschke product of degree at most $n-1$ (in which case $F(p) = c$ for all p with n zeros in D), or that $F(q) > F(p)$ for some polynomial q of degree n with zeros in G near the zeros of p. Let b be the Blaschke product with zero sequence $a_1, \ldots, a_n$ and let $d = F(p) = \operatorname{dist}(h, bH^\infty)$. We may assume that h is not identically zero. If $d = 0$, just perturb a_n by a small amount to move it away from the zero set of h while still remaining in G. For

$d > 0$, there exists a unique Blaschke product B with at most $m \le n-1$ zeros which interpolates the function h/d along the a_j. If $m < n-1$, then we also have, by Proposition 3, that

$$d = \operatorname{dist}\left(h, \left[\prod_{k=n-m}^{n} (z-a_k)\right] H^\infty\right).$$

By the inductive assumption there exists a polynomial p_1 whose zeros lie in G and such that d is less than $\operatorname{dist}(h, p_1 H^\infty)$. Now let q be a polynomial obtained from p_1 by adjoining $n-m-1$ zeros in G. Then $F(q) \ge F(p_1) > F(p)$. If $m = n-1$, we have for each small $s > 0$, a Blaschke product B_s of order $n-1$ which interpolates $h/(d+s)$ along $a_1, \dots, a_{n-1}$ and such that $\|B - B_s\|_\infty \to 0$ as $s \to 0$. Let $f_s = B_s - h/(d+s)$. Then $f_s \to B - h/d$ as $s \to 0$. If $a_k \ne a_{k+1} = \cdots = a_n$, then either $h = dB$ or, by Hurwitz' Theorem, some f_s has $n-k$ zeros (counting multiplicities) in G near a_n. Denote them by $a^{\#}_{k+1}, \dots, a^{\#}_n$. Letting q be a polynomial of degree n with zero set $\{a_1, \dots, a_k, a^{\#}_{k+1}, \dots, a^{\#}_n\}$, we have, by Proposition 3, that $F(q) = d + s > d$.

Finally, to relate the work in this paper to the operator-theoretic context of Pták and Young's original conjecture, we have (see [**2**]), as a corollary to Theorem 1,

THEOREM 2. *If $h \in H^\infty$ and $0 < r < 1$, then among all $n \times n$ contractions A with all eigenvalues in the disk $\{z\colon |z| \le r\}$, $\|h(A)\|$ attains its maximum at a matrix A having all of its eigenvalues on the circle $\{z\colon |z| = r\}$.*

ACKNOWLEDGMENTS. The author is grateful to Donald Sarason for several valuable conversations and to the referee for pointing out an inaccuracy in the original manuscript.

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