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REGULAR BOUNDARY ELEMENTS

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ABSTRACT. In a Banach algebra the regular elements in the closure of the group of invertibles are in a generalized sense "of index zero".

1. If A is a ring, with identity 1 and invertible group A^{-1}, or more generally **[6]** an additive category, we shall write

$$\overline{A} = \{a \in A \colon a \in aAa\} \tag{1.0.1}$$

for the *regular* or "relatively Fredholm" elements of A, and

$$A^{-1}\dot{A} = \dot{A}A^{-1} = \{a \in A \colon a \in aA^{-1}a\} \tag{1.0.2}$$

for the *decomposably regular* or "relatively Weyl" elements of A; these include the invertible elements A^{-1}, and also the *idempotents*

$$\dot{A} = \{a \in A \colon a^2 = a\}. \tag{1.0.3}$$

1.1 THEOREM. *If A is a Banach algebra then*

$$A^{-1}\dot{A} = \overline{A} \cap \mathrm{cl}(A^{-1}). \tag{1.1.1}$$

PROOF. The decomposably regular elements are always regular; if A is a normed algebra and if $a \in A$ is decomposably regular then we can write

$$a = cp = qc \quad \textit{with } c \in A^{-1},\ p \in \dot{A},\ q \in \dot{A}; \tag{1.1.2}$$

we have anticipated this in the notation for (1.0.2). Putting for each n

$$b_n = c(p + (1/n)(1-p)), \qquad b_n' = (p + n(1-p))c^{-1} \tag{1.1.3}$$

gives

$$\|a - b_n\| \to 0 \quad \textit{and} \quad b_n' b_n = 1 = b_n b_n', \tag{1.1.4}$$

so that also $a \in \mathrm{cl}(A^{-1})$. Conversely, without restriction on A, we claim that it is sufficient, for $a \in A$ to be decomposably regular, that there are $a' \in A$ and $b \in A$ for which

$$a = aa'a \quad \textit{and} \quad a' = a'aa' \quad \textit{and} \quad b \in A^{-1} \quad \textit{and} \quad 1 + (b-a)a' \in A^{-1}. \tag{1.1.5}$$

Indeed if (1.1.5) holds and we define

$$a'' = a' + (1 - a'a)b^{-1}(1 - aa') \tag{1.1.6}$$

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then a'' is an invertible generalised inverse for a. To see this write

(1.1.7) $$p = a'a, \qquad q = aa',$$

$$k' = (1 + (b - a)a')^{-1}, \qquad k'' = 1 - a'k'(b - a) = (1 + a'(b - a))^{-1};$$

then

(1.1.8)
$$a = aa''a \quad \textit{and} \quad (a + (1 - q)k'b(1 - p))a'' = 1 = a''(a + (1 - q)bk''(1 - p)),$$

since

(1.1.9) $$k'bp = a = qbk'' \quad \textit{and} \quad k'(1 - q) = 1 - q \quad \textit{and} \quad (1 - p)k'' = 1 - p.$$

The argument that (1.1.5) and (1.1.6) imply (1.1.8) works in an arbitrary ring or an additive category; if we specialise to a Banach algebra then we can ensure the last part of (1.1.5) by asking that

(1.1.10) $$\|b - a\| \, \|a'\| < 1. \quad \square$$

2. The extreme simplicity of the statement and proof of Theorem 1.1 conceals a long gestation and a difficult birth. In 1977 Treese and Kelly [**8**] claimed a slightly stronger result for the algebra A of operators on a Banach space X. Unfortunately their argument fails; recently Gonzalez [**3**] has found a counterexample. Gonzalez also gives a proof of Theorem 1.1 for operators; his argument rests on a theorem of Caradus [**1**, Chapter 5, Theorem 13] involving two kinds of "gap" between subspaces. Part of the proof of Caradus' theorem is taken from Kato [**7**, Chapter IV]; unfortunately there is a subtle difference between the definitions given by Caradus and by Kato. The argument we have given for Theorem 1.1 was, however, discovered by following the trail marked out by Gonzalez [**3**]. It is clear that the result extends to those possibly incomplete algebras [**2**] for which the invertible group is open.

When A is the algebra of operators on a Banach space and $a \in A$ is a Fredholm operator then the conclusion of Theorem 1.1 follows from the continuity of the *index*. More generally, if T: $A \to B$ is a homomorphism of rings, or an additive functor between categories, there is inclusion [**5**, **6**].

(2.0.1) $$A^{-1} \subseteq A^{-1} + T^{-1}(0) \subseteq T^{-1}(B^{-1}).$$

We shall call $T^{-1}(B^{-1})$ the *T-Fredholm* elements of A and $A^{-1} + T^{-1}(0)$ the *T-Weyl* elements. If A is the algebra of bounded operators and T takes the quotient by the ideal of compact operators then these are, respectively, the Fredholm operators and the Fredholm operators of index zero; if instead A is the continuous functions on the disc and T restricts to the circle then the "Fredholm" functions do not vanish on the circle, and for the "Weyl" functions the "winding number" or *topological degree* of the boundary value function is zero. The continuity of the index in each case predicts that the "Weyl" elements should be open and closed in the "Fredholm" elements.

2.1 THEOREM. *If T: $A \to B$ is a continuous homomorphism of Banach algebras which is decomposably regular in the sense that*

(2.1.1) $$T^{-1}(B^{-1}) \subseteq \overline{A} \quad \textit{and} \quad 1 + T^{-1}(0) \subseteq A^{-1}\dot{A},$$

then $A^{-1} + T^{-1}(0)$ *is open and closed in* $T^{-1}(B^{-1})$:

(2.1.2) $$A^{-1} + T^{-1}(0) = \mathrm{int}(A^{-1} + T^{-1}(0)) = T^{-1}(B^{-1}) \cap \mathrm{cl}(A^{-1}).$$

PROOF. If A^{-1} is open in A then so is $A^{-1} + T^{-1}(0)$. More generally, in an additive category A we have

(2.1.3)
$$1 - a'a \in T^{-1}(0), \quad a' \in A^{-1}, \quad 1 + a'(b-a) \in A^{-1} \Rightarrow b \in A^{-1} + T^{-1}(0).$$

For the second part of (2.1.2) we combine (1.1.1) with

(2.1.4) $$A^{-1} + T^{-1}(0) = T^{-1}(B^{-1}) \cap A^{-1}\mathring{A},$$

which is a consequence of the decomposable regularity (2.1.1). Forward inclusion in (2.1.4) follows at once from (2.1.1); conversely if $a = aa'a$ with a' invertible in A and $T(a)$ invertible in B then

(2.1.5) $$1 - a'a \in T^{-1}(0) \quad \textit{and} \quad a = (a')^{-1} + (a')^{-1}(1 - a'a). \quad \square$$

The decomposable regularity condition (2.1.1) fails if T: $A \to B$ is restriction to the circle for continuous functions on the disc. In this case (2.1.2) still holds, but for a different reason: T is onto and the invertible group A^{-1} is connected. If we write **[4]**

(2.1.6) $$\mathrm{Exp}(A) = \{e^{c_1}e^{c_2} \cdots e^{c_k} : k \in \mathbb{N}, c \in A^k\}$$

for the connected component of the identity in the topological group A^{-1}, then we are assuming

(2.1.7) $$T(A) = B \quad \textit{and} \quad A^{-1} = \mathrm{Exp}(A).$$

But now [**4**, Theorem 2]

(2.1.8) $$A^{-1} + T^{-1}(0) = \mathrm{Exp}(A) + T^{-1}(0) = T^{-1}(\mathrm{Exp}(B)),$$

and the right-hand side is both open and closed in $T^{-1}(B^{-1})$.

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