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A SYMBOLIC CALCULUS FOR ANALYTIC CARLEMAN CLASSES

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ABSTRACT. Let $\mathscr{C}_M(I_\alpha)$ be the analytic Carleman class of $\mathscr{C}^\infty$-functions f defined in a sector $I_\alpha = \{z \in \mathbf{C}\colon |\arg z| \leqslant \alpha\pi/2\} \cup \{0\}$ $(0 \leqslant \alpha \leqslant 1)$ and analytic in its interior such that $\|f^{(n)}\|_\infty \leqslant C\lambda^n M_n$ $(n \geqslant 0)$, $C = C(f)$, $\lambda = \lambda(f)$. In this paper, we give necessary and sufficient conditions in order that $\mathscr{C}_M(I_\alpha)$ be inverse-closed. As a corollary, we obtain a characterization of $\mathscr{C}_M(\mathbf{R}_+)$ as an inverse-closed algebra, thus establishing the converse of a theorem of Malliavin [4] for the half-line.

1. Given a sequence $M = \{M_n\}$ of positive numbers, we say that the sequence $\{A_n\}$, where $A_n = (M_n/n!)^{1/n}$, is almost increasing if there exists a positive constant K such that for all m and n with $n \leqslant m$, $A_n \leqslant KA_m$. P. Malliavin [**4**] proved that if M is log-convex and the associated sequence $A = \{A_n\}$ is almost increasing, then the algebra $\mathscr{C}_M(I)$ of $\mathscr{C}^\infty$-functions on a linear interval I with

$$\|f^{(n)}\|_\infty \leqslant C\lambda^n M_n \quad (n \geqslant 0), \qquad C = C(f), \qquad \lambda = \lambda(f),$$

is inverse closed, i.e., if $f \in \mathscr{C}_M(I)$ and $f(x) \neq 0$ for all $x \in I$, then $f^{-1} \in \mathscr{C}_M(I)$. The problem whether the converse is true, viz., if $\mathscr{C}_M(I)$ is inverse closed then the sequence A is almost increasing, was taken up by W. Rudin [**6**] who proved that it is so if $\mathscr{C}_M(R)$ is a non-quasi-analytic algebra of 2π-periodic functions. Subsequently J. Boman and L. Hörmander [**1**] proved the same result for arbitrary classes $\mathscr{C}_M(\mathbf{R})$ by a long and ingenious method. The problem whether the converse of Malliavin's theorem is true for classes $\mathscr{C}_M(I)$, where I is a half-line or a finite interval, remains unsolved. Recently J. Bruna [**2**] studied the related problem for Beurling classes.

In this paper we give a necessary and sufficient condition in order that the algebra $\mathscr{C}_M(I_\alpha)$ of $\mathscr{C}^\infty$-functions f in a sector $I_\alpha = \{z \in \mathbf{C}\colon |\arg z| \leqslant \alpha\pi/2\} \cup \{0\}$ $(0 \leqslant \alpha \leqslant 1)$ and analytic in its interior such that $\|f^{(n)}\|_\infty \leqslant C\lambda^n M_n$, $C = C(f)$, $\lambda = \lambda(f)$, be inverse-closed. As a corollary we obtain a characterization of the inverse-closed algebra $\mathscr{C}_M(\mathbf{R}_+)$, thus establishing the converse of Malliavin's theorem for the half-line.

2. The analytic Carleman classes $\mathscr{C}_M(I_\alpha)$ have been extensively studied by several authors, notably by B. I. Korenbljum [**3**] who gave conditions for the nontriviality and the quasi-analyticity of these classes.

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Given a sequence $M = \{M_n\}$ of positive numbers, we define the sequence $M^\alpha = \{M_n^\alpha\}$ by the rule: $N_n = n^{(1-\alpha)n}M_n$, $N_n^c = n^{(1-\alpha)n}M_n^\alpha$, where $\{\log N_n^c\}$ is the largest convex minorant of the sequence $\{\log N_n\}$ (cf. S. Mandelbrojt [5]). The class $\mathscr{C}_M(I_\alpha)$ coincides with the class $\mathscr{C}_{M^\alpha}(I_\alpha)$ (cf. for example J. A. Siddiqi and A. El Koutri [7, Theorem 1]). Using Leibnitz' formula and the fact that $\{\log N_n^c\}$ is convex, it can be easily seen that $\mathscr{C}_{M^\alpha}(I_\alpha)$ is an algebra.

Our main result is as follows:

THEOREM 1. *The following assertions are equivalent*:

(a) *The sequence* $A^\alpha = \{A_n^\alpha\}$, *where* $A_n^\alpha = (M_n^\alpha/n!)^{1/n}$, *is almost increasing.*

(b) *If* $f \in \mathscr{C}_M(I_\alpha)$ *and* g *is analytic in a domain containing the range of* f, *then* $g \circ f \in \mathscr{C}_M(I_\alpha)$.

(c) $\mathscr{C}_M(I_\alpha)$ *is inverse-closed.*

PROOF. That (a) implies (b) follows directly from the formula of Faà di Bruno, viz.,

$$(g \circ f)^{(n)}(z) = \sum_{\substack{k_1+k_2+\cdots+k_n=k \\ k_1+2k_2+\cdots+nk_n=n}} \frac{n!}{k_1!k_2!\cdots k_n!} g^{(k)}(f(z)) \left(\frac{f'(z)}{1!}\right)^{k_1} \cdots \left(\frac{f^{(n)}(z)}{n!}\right)^{k_n}.$$

Trivially (b) implies (c). We now show that (c) implies (a). Put $N_n^c = n^{(1-\alpha)n}M_n^\alpha$. The sequence $\{N_n^c\}$ is log-convex in view of the regularization procedure adopted for obtaining $\{M_n^\alpha\}$. Let

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{2^k} N_k^c \left(\frac{N_{k+1}^c}{N_k^c}\right)^{-k} h\left(\frac{N_{k+1}^c}{N_k^c} z\right),$$

where

$$h(z) = \sum_{\nu=0} \frac{(-z)^\nu}{[(\nu+1)(2-\alpha)+1]!} \qquad (x! = \Gamma(x+1)).$$

Since

$$|h^{(n)}(z)| \leqslant 2\frac{n!e^n}{n^{(2-\alpha)n}}$$

and

$$(N_{k+1}^c/N_k^c)^{n-k} \leqslant N_n^c/N_k^c$$

for all k, because of the fact that $\{N_n^c\}$ is log-convex, we have

$$|f^{(n)}(z)| \leqslant \sum_{k=0}^{\infty} \frac{1}{2^k} N_k^c \left(\frac{N_{k+1}^c}{N_k^c}\right)^{n-k} \left|h^{(n)}\left(\frac{N_{k+1}^c}{N_k^c} z\right)\right| \leqslant 4e^n M_n^\alpha$$

for all $z \in I_\alpha$. Moreover

$$f^{(n)}(0) = \left\{ \sum_{k=0}^{\infty} \frac{1}{2^k} N_k^c \left(\frac{N_{k+1}^c}{N_k^c} \right)^{n-k} \right\} \frac{n!(-1)^n}{[(n+1)(2-\alpha)+1]!}$$
$$= t_n \frac{n!(-1)^n}{[(n+1)(2-\alpha)+1]!},$$

where $t_n \geqslant N_n^c/2^n$.

Let $g(z) = 1/(\lambda - z)$, where $\lambda > M_0^\alpha$. Since $\lambda - f \in \mathscr{C}_{M^\alpha}(I_\alpha)$ and $\mathscr{C}_{M^\alpha}(I_\alpha)$ is inverse-closed, it follows that

$$g \circ f = (\lambda - f)^{-1} \in \mathscr{C}_{M^\alpha}(I).$$

We, therefore, have

$$\left|(g \circ f)^{(n)}(0)\right|$$
$$= \left| \sum_{\substack{k_1 + k_2 + \cdots + k_n = k \\ k_1 + 2k_2 + \cdots + nk_n = n}} \frac{n!}{k_1! \cdots k_n!} \frac{k!}{(\lambda - f(0))^{k+1}} \left(\frac{f'(0)}{1!} \right)^{k_1} \cdots \left(\frac{f^{(n)}(0)}{n!} \right)^{k_n} \right|$$
$$\leqslant C\lambda^n M_n^\alpha$$

so that

$$\sum_{\substack{k_1 + k_2 + \cdots + k_n = k \\ k_1 + 2k_2 + \cdots + nk_n = n}} \frac{n!}{k_1! \cdots k_n!} \frac{k!}{(\lambda - f(0))^{k+1}} \left(\frac{t_1}{(2\gamma + 1)!} \right)^{k_1} \cdots \left(\frac{t_n}{((n+1)\gamma + 1)!} \right)^{k_n} \leqslant C\lambda^n M_n^\alpha,$$

where $\gamma = 1 - \alpha$.

If we choose $k_l = k$, $lk = n$, $k_1 = \cdots = k_{l-1} = k_{l+1} = \cdots = k_n = 0$, then we have

$$\left(\frac{t_l}{((l+1)\gamma + 1)!} \right)^k \leqslant C_1 \lambda_1^n \frac{M_n^\alpha}{n!}$$

or

$$\left(\frac{M_l^\alpha}{l!} \right)^{1/l} \leqslant K_1 \left(\frac{M_n^\alpha}{n!} \right)^{1/n}.$$

If n is not a multiple of l, let $lm \leqslant n \leqslant l(m+1)$ so that, using Stirling formula and the fact that $\{(N_n^c)^{1/n}\}$ is increasing, we get

$$\left(\frac{M_n^\alpha}{n!} \right)^{1/n} = \frac{\left(n^{(1-\alpha)n} M_n^\alpha\right)^{1/n}}{(n!)^{1/n} n^{1-\alpha}} \geqslant \frac{\left((ml)^{(1-\alpha)ml} M_{ml}^\alpha\right)^{1/ml}}{(n!)^{1/n} n^{1-\alpha}} \geqslant \frac{1}{2^{2-\alpha} K_1 e} \cdot \left(\frac{M_l^\alpha}{l!} \right)^{1/l}.$$

Thus for $n \geqslant l$

$$\left(\frac{M_l^\alpha}{l!} \right)^{1/l} \leqslant K \left(\frac{M_n^\alpha}{n!} \right)^{1/n},$$

where K is independent of l and n and this completes the proof of the theorem.

If $\alpha = 0$, then in the notation of S. Mandelbrojt (cf. [5, p. 227]) $M_n^0 = M_n^d$ for all $n \geqslant 1$ and Theorem 1 yields the following characterization of the inverse-closed algebra $\mathscr{C}_M(\mathbf{R}_+)$.

THEOREM 2. *The following assertions are equivalent*:

(a) *The sequence* $A^d = \{A_n^d\}$, *where* $A_n^d = (M_n^d/n!)$, *is almost increasing.*

(b) *If* $f \in \mathscr{C}_M(\mathbf{R}_+)$ *and* g *is analytic in a domain containing the range of* f, *then* $g \circ f \in \mathscr{C}_M(\mathbf{R}_+)$.

(c) $\mathscr{C}_M(\mathbf{R}_+)$ *is inverse-closed.*

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