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ON THE FREE GENUS OF KNOTS

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ABSTRACT. The class of knots consisting of twisted Whitehead doubles can have arbitrarily large free genus but all have genus 1.

1. Introduction. Let $K \subset S^3$ be a knot. The free genus of K denoted by $g_f(K)$ is the minimal genus of all Seifert surfaces S for K such that $\pi_1(S^3 - \overset{\circ}{N}(S))$ is free. Let $g(K)$ denote the genus of K. Note that $g_f(K) \geq g(K)$ for all knots. It is conjectured (see [**1**, Problem 1.20a]) that the difference between the free genus and the genus can be arbitrarily large. In this paper we show that the conjecture is true by considering twisted Whitehead doubles.[1] The conjecture will follow from the following theorem.

THEOREM 1. *Let $D_k(K)$, $k \neq 0$, denote the Whitehead double with k twists of a knot $K \subset S^3$. Suppose that*

$$\operatorname{rank} H_1(B_{|4k+1|}(K)) = r$$

where $\operatorname{rank}(G)$ *is the minimal number of generators for G and $B_n(K)$ is the n-fold cyclic cover of S^3 branched over K. Then*

$$g_f(D_k(K)) \geq \frac{2r-1}{2|4k+1|}.$$

Note. $g(D_k(K)) = 1$ for all knots K and all $k \in \mathbf{Z}$. The conjecture follows from the theorem by the following example:

Let K^n be the connected sum of n trefoils and let $k = -1$. Then $|4k+1| = 3$ and rank $H_1(B_3(K^n)) = 2n$. From the theorem it follows that $g_f(D_{-1}(K^n)) \geq (4n-1)/6$ while $g(D_{-1}(K^n)) = 1$. Let $n \to \infty$ and the conjecture follows.

I would like to thank Professor C. Mc A. Gordon for many useful conversations concerning this work.

2. Terminology and preliminary lemmas. Let $K \subset S^3$ be an unoriented knot in S^3. $N(K)$ will denote a tubular neighborhood of K. The exterior of K is $X = S^3 - \overset{\circ}{N}(K)$. Let $f\colon (S^1 \times D^2) \to N(K)$ be the 0 framing homeomorphism that is $[f(S^1 \times *)] = 0 \in H_1(X)$ for $* \in \partial D^2$. Choosing an orientation for K determines a longitude meridian pair $\lambda, \mu \in H_1(\partial X)$ as follows:

$$\lambda = [f(S^1 \times *)] \quad \text{where } * \in \partial D^2 \text{ and } \mu = [f(* \times \partial D^2)], \qquad * \in S^1.$$

Let $r = m/n$ where $m, n \in \mathbf{Z}$, $n = 0$ is allowed.

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[1] It is still unknown for untwisted doubles as suggested in [**1**, Problem 1.20a].

$L_1 =$

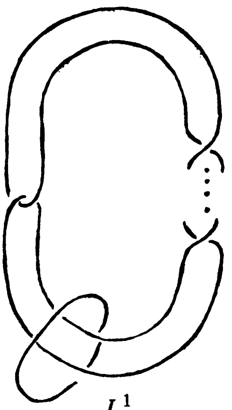

FIGURE 1

The manifold obtained by doing Dehn surgery of type r on K is $X \cup_h S^1 \times D^2$ where $h: S^1 \times \partial D^2 \to \partial X$ is a homeomorphism such that $[h(* \times \partial D^2)] = m\mu + n\lambda \in H_1(\partial X)$.

We need the following three lemmas.

LEMMA 1. *Let $K \subset S^3$ be a knot. Let $X = S^3 - \mathring{N}(K)$ and let $g_f(K)$ denote the minimal genus of all unknotted Seifert surfaces for K. Let* $\operatorname{rank} \pi_1(X)$ *denote the minimal number of generators for $\pi_1(X)$. Then*

$$\operatorname{rank} \pi_1(X) \leq g_f(K) + 1.$$

LEMMA 2. *Let G be a finitely generated group and $H \subset G$ be a subgroup of finite index k in G. Then*

$$\operatorname{rank}(G) \geq \frac{\operatorname{rank}(H) - 1}{k} + 1.$$

The proofs of the above lemmas will appear in §4.

LEMMA 3. $H_1(B_2(D_k(K))) \cong \mathbf{Z}_{|4k+1|}$.

REMARK. This can be proved easily by computing the Seifert matrices for $D_k(K)$. We will prove it via the following construction which is needed for the main theorem.

PROOF. Fix an orientation for K. Now we have a well-defined longitude meridian pair λ, μ in $H_1(\partial X) = H_1(\partial(S^3 - \mathring{N}(K)))$. We can think of $S^3 - \mathring{N}(D_k(K))$ as the complement in S^3 of the link L_1 with $X = S^3 - \mathring{N}(K)$ glued to the boundary component corresponding to L_1^1 (see Figure 1).

The gluing map h for ∂X is given by

$$\mu_x \xrightarrow{h_*} 0 \cdot \mu_{L_1^1} + 1 \cdot \lambda_{L_1^1}, \quad \lambda_x \xrightarrow{h_*} 1 \cdot \mu_{L_1^1} + 0 \cdot \lambda_{L_1^1}$$

or in the matrix formed by $h_* = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. We can twist about L_1^1 $-k$ times and use the symmetry of the Whitehead link to get the following description of $S^3 - \mathring{N}(D_k(K))$

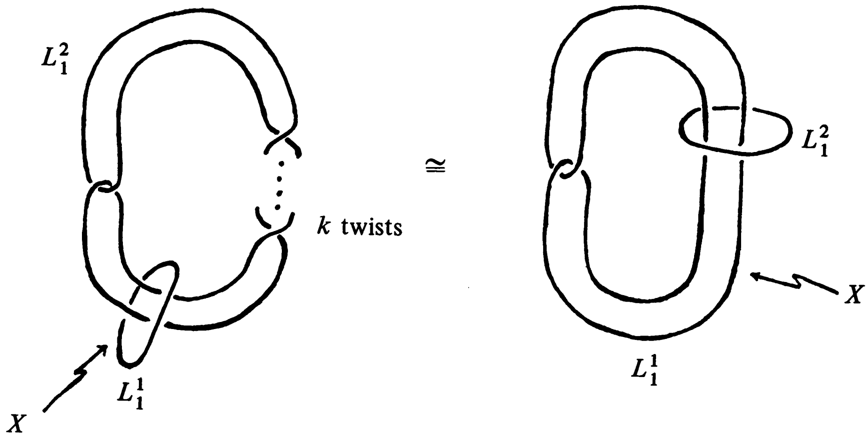

FIGURE 2

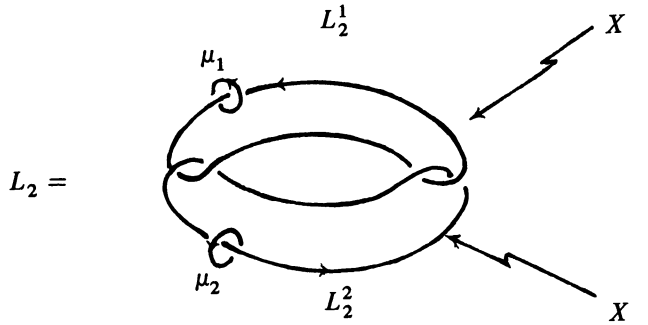

FIGURE 3

(see Figure 2) where the surgery instructions for ∂X are given by the map

$$h_* = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(see [**2**, pp. 264–267]).

Let i be the inclusion map $i\colon X \to S^3 - \overset{\circ}{N}(D_k(K))$. A meridian of X is mapped by i to a longitude of L_1^1 which is null homologous in the complement of $D_k(K)$. Since $\pi_1(X)$ is generated by meridians, it follows that X lifts to the branched double cover of S^3 branched along $D_k(K)$. In other words $B_2(D_k(K))$ can be thought of as the 3-manifold obtained by gluing two copies of $X = S^3 - \overset{\circ}{N}(K)$, one to each boundary component, of the complement in S^3 of the above link L_2 (see Figure 3).

In terms of the obvious longitude meridian pair $\lambda_i \mu_i$, $i = 1, 2$ of each unknotted components L_2^1 and L_2^2 of L_2, a longitude of L_1^1 lifts to $1 \cdot \lambda_i - 2 \cdot \mu_i$. Hence, the gluing map for ∂X is given by the matrix

$$h_* = \begin{bmatrix} 2k+1 & -2 \\ -k & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Note that $H_1(S^3 - \overset{\circ}{N}(L_2)) \cong \mathbf{Z}_{\mu_1} \oplus \mathbf{Z}_{\mu_2}$. A longitude of the unknotted component L_2^1 is homologous to $2\mu_2$. Therefore, because of the symmetry between L_2^1, L_2^2 gluing the two copies of X gives us the following relations:

$$(2k+1)\mu_1 - 2k\mu_2 = 0, \quad -2k\mu_1 + (2k+1)\mu_2 = 0.$$

The relation matrix

$$\begin{pmatrix} 2k+1 & -2k \\ -2k & 2k+1 \end{pmatrix}$$

is equivalent to the matrix $(4k+1)$ so $H_1(B_2(D_k(K)))$ is a cyclic group of order $|4k+1|$ generated by $\bar{\mu}_1$, the image of μ_1 in $\mathbf{Z}_{|4k+1|}$.

3. Proof of the main theorem. Let $W_n(D_k(K))$ denote the n-fold cyclic cover of $S^3 - \overset{\circ}{N}(D_k(K))$. Recall that $H_1(W_n) = H_1(B_n(D_k(K))) \oplus \mathbf{Z}$. By Lemma 3 we have a map $\varphi: \pi_1(W_2(D_k(K))) \to \mathbf{Z}_{|4k+1|}$. Let W denote the $|4k+1|$ cyclic cover of $W_2(D_k(K))$ corresponding to φ, hence $\pi_1(W)$ is a subgroup of index $2 \cdot |4k+1|$ of $\pi_1(S^3 - D_k(K))$. By Lemma 2

$$\operatorname{rank} \pi_1(S^3 - D_k(K)) \geq \frac{\operatorname{rank} \pi_1(W) - 1}{2 \cdot |4k+1|} + 1.$$

Let Y denote the $|4k+1|$ cyclic cover of $B_2(D_k(K))$. Note that $\operatorname{rank} \pi_1(W) \geq \operatorname{rank} H_1(W) \geq \operatorname{rank} H_1(Y)$. Suppose that $\operatorname{rank} H_1(Y) = 2r$; then by Lemma 1

$$g_f(D_k(K)) \geq \operatorname{rank} \pi_1(S^3 - D_k(K)) - 1 \geq \frac{2r-1}{2 \cdot |4k+1|}.$$

Therefore in order to complete the proof it is sufficient to show that if

$$\operatorname{rank} H_1(B_{|4k+1|}(K)) = r$$

then $\operatorname{rank} H_1(Y) = 2r$.

REMARK. If we choose to consider $\operatorname{rank} H_1(W)$ instead of $\operatorname{rank} H_1(Y)$ we can obtain a slightly better formula, i.e.,

$$g_f(D_k(K)) \geq \frac{2r - 1 + |4k+1|}{2 \cdot |4k+1|}.$$

We omit the details.

From the proof of Lemma 3, we see that the image of a meridian μ of X under the map i_* induced by the inclusion $i: X \hookrightarrow B_2(D_k(K))$ is $2\bar{\mu}_1$. Since $(4k+1, 2) = 1$, μ is mapped onto a generator of $H_1(B_2(D_k(K)))$. If $i_*(n\mu) = 0$ then $n = (4k+1)m$ and $i_*(|4k+1|\mu) = 0$. Hence the $|4k+1|$ cyclic cover of X lifts to Y and in Y we have exactly two copies of the $|4k+1|$ cyclic cover of X.

Following [**1**, pp. 264–267] we have description for $B_2(D_k(K))$ as in Figure 4 (r_0 will denote the surgery instructions for L^0).

The twist homeomorphism about L^0 changes the gluing map h for X to

$$h = \begin{pmatrix} 1 & 0 \\ -k & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

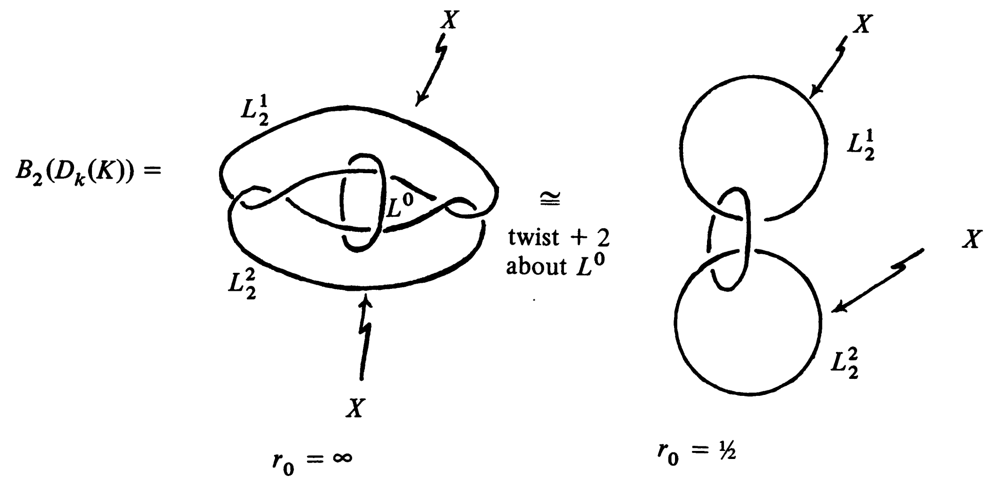

FIGURE 4

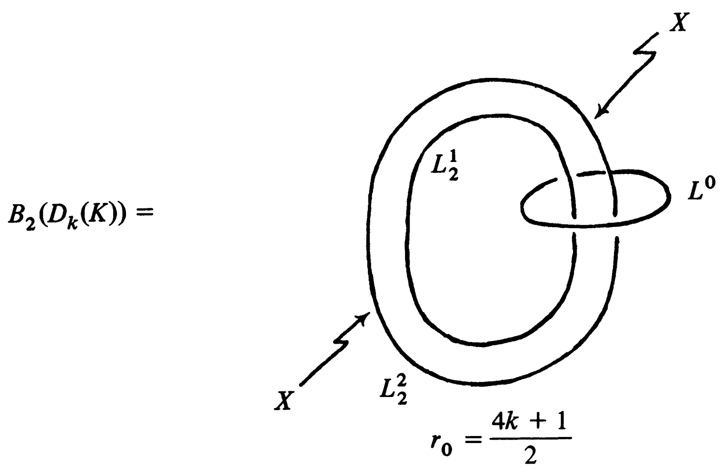

FIGURE 5

Now twist $+k$ times about L_2^1 and $+k$ times about L_2^2 to get Figure 5 with gluing map h given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

That is $B_2(D_k(K))$ is the lens space $L_{(|4k+1|,2)}$ with two solid tori removed and two copies of X glued in. Let $\tilde{X}_{|4k+1|}$ denote the $|4k+1|$ cyclic cover of X. The $|4k+1|$ cyclic cover of $(L_{(|4k+1|,2)} - \overset{\circ}{N}(L_2))$ is $S^3 - 2$ tori. Hence Y is the complement in S^3 of the two-component link $\tilde{L}$ (see Figure 6) with two copies of $\tilde{X}_{|4k+1|}$ glued to the boundary components of $\tilde{L}$.

In terms of the obvious longitude meridian pair for each unknotted component $\tilde{L}_i$ the gluing map h is given by $h = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Note that the surgery coefficient $r_0 = |4k+1|/2$ has lifted to $\tilde{r}_0 = 1/2$.

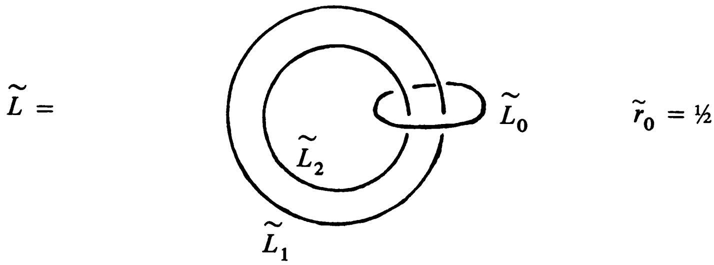

FIGURE 6

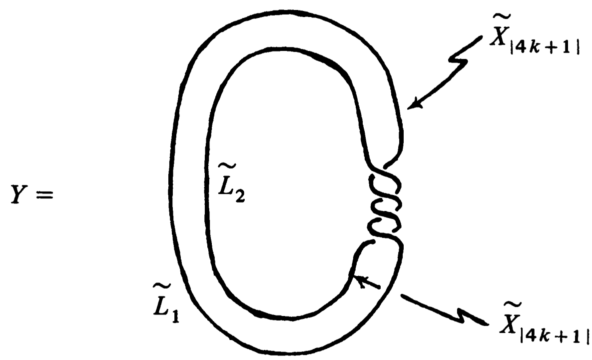

FIGURE 7

After twisting -2 times about $\tilde{L}_0$ we get Figure 7 and

$$h = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Recall that $H_1(\tilde{X}_{|4k+1|}) = \mathbf{Z} \oplus H_1(B_{|4k+1|}(K))$ where $\mathbf{Z}$ is generated by $\tilde{\mu}$ the lift of a meridian μ of X.

h maps $\tilde{\mu}$ to a $-2/1$ curve on the boundary components of $S^3 - \tilde{L}$. The $-2/1$ curve is null homologous in $S^3 - \tilde{L}$ and it follows by the Mayer Vietoris sequence that

$$H_1(Y) = H_1(S^3 - \tilde{L} \cup_{h_1} \tilde{X}_{|4k+1|} \cup_{h_2} \tilde{X}_{|4k+1|}) \cong H_1(B_{|4k+1|}(K)) \oplus H_1(B_{|4k+1|}(K))$$

by the assumption $\operatorname{rank} H_1(B_{|4k+1|}(K)) = r$. So $\operatorname{rank}(H_1(Y)) = 2r$ and the proof is complete.

Let K be a knot in S^3 and let $t(K)$ denote the tunnel number of K, that is the minimal number of 2 handles that need be removed from $S^3 - \mathring{N}(K)$ so that the complement is a handlebody. $g(K)$ as before denotes the genus of K.

COROLLARY. *$t(K) - g(K)$ can be arbitrarily large.*

PROOF. $\pi_1(S^3 - K)$ has a presentation with $t(K) + 1$ generators and $t(K)$ relators so $\operatorname{rank} \pi_1(S^3 - K) \leq t(K) + 1$. Let K^n be the connected sum of n trefoils and set $k = -1$, $g(D_{-1}(K^n)) = 1$ and $\operatorname{rank} \pi_1(S^3 - D_{-1}(K^n)) \geq (4n - 1)/6$ so $t(D_{-1}(K^n)) \geq (4n - 7)/6$. Now set n arbitrarily large.

4. Proof of lemmas.

PROOF OF LEMMA 1. Let S be a Seifert surface for K which realizes the free genus. Let $\overset{\circ}{N}(S)$ be a product neighborhood of S in $S^3 - \overset{\circ}{N}(K)$. $S^3 - \overset{\circ}{N}(S)$ is a handlebody of genus $g_f(K)$. Hence $\pi_1(S^3 - \overset{\circ}{N}(K))$ is an HNN extension of the free group with $g_f(K)$ generators and has a presentation with $g_f(K)+1$ generators [**3**, p. 180]. So $\operatorname{rank} \pi_1(S^3 - \overset{\circ}{N}(K)) \leq g_f(K) + 1$.

PROOF OF LEMMA 2. Let $\langle X_1, \ldots, X_g | R_1, \ldots, R_l\rangle$ be a presentation for G with g minimal. Construct a 2 complex K with 1 vertex g edges and l faces such that $\pi_1(K) = G$. K has a k fold cover $\tilde{K}$ such that $\pi_1(\tilde{K}) = H$. $\tilde{K}$ has k vertices, $k \cdot g$ edges and $k \cdot l$ faces. Shrink a maximal tree in the 1 skeleton of $\tilde{K}$.

Set a base point for $\pi_1(\tilde{K})$ at the remaining vertex. We have a 2 complex with 1 vertex, $kg - k + 1$ edges and kl faces. This gives rise to a presentation for H with $kg - k + 1$ generators. If h is the minimal number of generators in any presentation for H then

$$kg - k + 1 \geq h \quad \text{so} \quad g \geq (h-1)/k + 1.$$

REFERENCES

1. R. Kirby, *Problems in low dimensional manifold theory*, Proc. Sympos. Pure Math., vol. 32, part 2, Amer. Math. Soc., Providence, R.I., 1978, pp. 273–312.
2. D. Rolfsen, *Knots and links*, Publish or Perish, Boston, Mass., 1976.
3. R. Lyndon and P. Schupp, *Combinatorial group theory*, Springer-Verlag, 1977.

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