GROUP RING IDEMPOTENTS WITH MINIMAL TRACE

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ABSTRACT. Let G be a group and let K be a field of characteristic 0. Let $e \neq 0$ be an idempotent of the group ring $K(G)$. It is shown that the trace of e attains its lower bound if and only if the support of e has a specific form.

1. Introduction. Let G be a group, let K be a field of characteristic 0 and let $K(G)$ be the group ring of G over K . For $\alpha = \sum_{x \in G} \alpha_x x \in K(G)$, the support of α , denoted by $\text{Supp}(\alpha)$, is the finite set $\{x: \alpha_x \neq 0\}$ and the trace of α , denoted by $\text{tr}(\alpha)$, is the coefficient α_1 . Let $e \neq 0$ be an idempotent of $K(G)$ and suppose $|\text{Supp}(e)| = n$. Using an inequality of D. S. Passman [3, Lemma 1.1.7], D. Farkas has shown that $\text{tr}(e) \geq 1/n$ [2, Remark 2.6]. We here give a more direct derivation of this second inequality and show that equality holds if and only if $\text{Supp}(e)$ is a subgroup and e is a primitive central idempotent of $K(\text{Supp}(e))$ corresponding to a linear character of $\text{Supp}(e)$ (Theorem 2.5).

2. Results. Throughout we follow the notation of [3] and assume, without loss, that K is a subfield of the complex numbers.

We begin with a theorem that characterizes for a finite group S the idempotent of the type we wish to consider for G an infinite group.

THEOREM 2.1. *Let S be a finite group of order n . Let $e \neq 0$ be an idempotent of $K(S)$. Then $\text{tr}(e) = 1/n$ if and only if e is the primitive, central idempotent associated with a linear character $\lambda: S \rightarrow K$, namely*

$$e = \frac{1}{n} \sum_{x \in S} \overline{\lambda(x)} x.$$

PROOF. (i) Let $\lambda: S \rightarrow K$ be a linear character and let

$$e = \frac{1}{n} \sum_{x \in S} \overline{\lambda(x)} x.$$

Then e is the primitive, central idempotent associated with λ [1, Lemma IV 7.1] and $\text{tr}(e) = 1/n$.

(ii) Suppose $\text{tr}(e) = 1/n$. By [2, Lemma 2.1.2(ii)]

$$\text{tr}(e) = \frac{\dim eK(S)}{n}.$$

Thus $\dim eK(S) = 1$ and so $eK(S)$ is an irreducible $K(S)$ -module. Let $\lambda: S \rightarrow K$ be the linear character corresponding to the module $eK(S)$. Let $e = \sum_{x \in S} e_x x$. Then for $y \in S$

$$ey^{-1} = \lambda(y^{-1})e,$$

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which implies that

$$\left(\sum_{x \in S} e_x x \right) y^{-1} = \lambda(y^{-1}) \left(\sum_{x \in S} e_x x \right).$$

Comparing the coefficient of 1 on both sides we have

$$e_y = \lambda(y^{-1})e_1 = \overline{\lambda(y)}1/n \quad (y \in S),$$

from which we obtain the result. $\square$

We now prove an elementary lemma from which we shall derive the aforementioned inequality, namely, $\text{tr}(e) \geq 1/n$.

LEMMA 2.2. *Let $\lambda_i \in \mathbf{R}$ ($i = 1, 2, \dots, n$). Then*

$$n \left(\sum_{i=1}^n \lambda_i^2 \right) - \left(\sum_{i=1}^n \lambda_i \right)^2 \geq 0$$

with equality if and only if $\lambda_1 = \lambda_2 = \dots = \lambda_n$.

PROOF.

$$\begin{aligned} n \left(\sum_{i=1}^n \lambda_i^2 \right) - \left(\sum_{i=1}^n \lambda_i \right)^2 &= (n-1) \left(\sum_{i=1}^n \lambda_i^2 \right) - 2 \left[\sum_{\substack{i,j \\ i < j}}^n \lambda_i \lambda_j \right] \\ &= \sum_{\substack{i,j \\ i < j}} (\lambda_i - \lambda_j)^2 \end{aligned}$$

from which the result follows. $\square$

COROLLARY 2.3. *Let $0 \neq \alpha = \sum_{x \in G} \alpha_x x \in K(G)$. Then*

$$\frac{\|\alpha\|^2}{|\alpha|^2} \geq \frac{1}{|\text{Supp}(\alpha)|}$$

with equality if and only if $|\alpha_x| = |\alpha_y|$ for all $x, y \in \text{Supp}(\alpha)$.

PROOF. Let $S = \text{Supp}(\alpha)$ and let $n = |\text{Supp}(e)|$. Then by definition

$$\|\alpha\|^2 = \sum_{x \in S} |\alpha_x|^2, \quad |\alpha| = \sum_{x \in S} |\alpha_x|,$$

so

$$n\|\alpha\|^2 - |\alpha|^2 = n \left[\sum_{x \in S} |\alpha_x|^2 \right] - \left[\sum_{x \in S} |\alpha_x| \right]^2.$$

Application of Lemma 2.2 yields the result. $\square$

COROLLARY 2.4 [2, REMARK 2.6]. *Let $0 \neq e = e^2 \in K(G)$. Then*

$$\text{tr}(e) \geq \frac{\|e\|^2}{|e|^2} \geq \frac{1}{|\text{Supp}(e)|}.$$

PROOF. This now follows immediately from Lemma 2.1.7 of [3]. $\square$

We now come to the main theorem of this paper.

THEOREM 2.5. *Let $e \neq 0$ be an idempotent of $K(G)$ such that*

$$\text{tr}(e) = 1/n,$$

where $n = |\text{Supp}(e)|$. Then $S = \text{Supp}(e)$ is a finite subgroup of G and there exists a linear character $\lambda: S \rightarrow K$ such that

$$e = \frac{1}{n} \sum_{x \in S} \overline{\lambda(x)} x.$$

PROOF. By Corollary 2.5

$$\frac{\|e\|^2}{|e|^2} = \frac{1}{|\text{Supp}(e)|} = \frac{1}{n}.$$

Thus, letting $e = \sum_{x \in G} e_x x$, we have, by Corollary 2.3,

$$|e_x| = |e_1| = e_1 = 1/n \quad (x \in S).$$

Let $y \in S$. Then, as $e^2 = e$, we have

$$e_y = \sum_{\substack{x, z \\ xz=y}} e_x e_z = \sum_{x \in G} e_x e_{x^{-1}y}.$$

Now for $x \in S$, $|e_x| = 1/n$ and so

$$|e_{x^{-1}y}| = \begin{cases} 1/n & (x^{-1}y \in S), \\ 0 & (x^{-1}y \notin S). \end{cases}$$

Thus

$$\begin{aligned} \frac{1}{n} = |e_y| &\leq \sum_{x \in G} |e_x| |e_{x^{-1}y}| \\ &= \sum_{x \in S} |e_x| |e_{x^{-1}y}| \leq \frac{1}{n} \sum_{x \in S} |e_{x^{-1}y}|. \end{aligned}$$

This is only possible if for all $x \in S$ $|e_{x^{-1}y}| = 1/n$ or, equivalently, $x^{-1}y \in S$. But y is arbitrary and so for all $x, y \in S$, $x^{-1}y \in S$ and hence S is a subgroup. The remaining assertions follow from Theorem 2.1. $\square$

REMARK 2.6. The conclusion above does not hold if we merely suppose that e is an idempotent for which

$$\frac{\|e\|^2}{|e|^2} = \frac{1}{|\text{Supp}(e)|}.$$

To see this let G be either the group $\langle x, y: y^{-1}xy = x^{-1}, y^2 = 1 \rangle$, which is the infinite dihedral group, or its homomorphic image $\langle x, y: y^{-1}xy = x^{-1}, y^2 = 1, x^3 = 1 \rangle$, which is the symmetric group of order 6. Then

$$e = \frac{(1 - x + x^{-1}) + (x - x^{-1} - 1)y}{2}$$

is a noncentral idempotent in $\mathbf{Q}(G)$, $\text{tr}(e) = \frac{1}{2}$, $|e| = 3$, $\|e\|^2 = \frac{3}{2}$ and

$$\frac{\|e\|^2}{|e|^2} = \frac{1}{6} = \frac{1}{|\text{Supp}(e)|}.$$

In either case G is generated by $\text{Supp}(e)$ and in the second case $G = \text{Supp}(e)$.

From our main theorem we obtain the following result which may be of independent interest.

THEOREM 2.7. Let $0 \neq e = e^2 \in K(G)$. Then

(i) $|e| \geq 1$, and

(ii) $|e| = 1$ if and only if $S = \text{Supp}(e)$ is a finite subgroup of G of order n and there exists a linear character $\lambda: S \rightarrow K$ such that

$$e = \frac{1}{n} \sum_{x \in S} \overline{\lambda(x)} x.$$

PROOF. By [3, Lemma 1.1.5(iii)] we have

$$\|e\| = \|ee\| \leq \|e\| |e|$$

and so $|e| \geq 1$.

(ii) We remark that if $S = \text{Supp}(e)$ is a finite subgroup of order n and e has the given form then it is an easy calculation that $|e| = 1$.

Suppose, conversely, that $|e| = 1$. Choose $m \in S$ such that

$$|e_m| = \text{Max}\{|e_x| : x \in S\}.$$

Then, as $e^2 = e$, we have $e_m = \sum_{u \in G} e_u e_{u^{-1}m}$ and hence

$$\begin{aligned} |e_m| &\leq \sum_{u \in G} |e_u| |e_{u^{-1}m}| \\ &= \sum_{u \in S} |e_u| |e_{u^{-1}m}| \leq \sum_{u \in S} |e_m| |e_{u^{-1}m}| \\ &\leq |e_m| \sum_{u \in S} |e_{u^{-1}m}| \leq |e_m| |e| \\ &= |e_m|. \end{aligned}$$

Thus we have equality throughout and so, in particular, for all $u \in S$

$$|e_u| = |e_m|.$$

Thus $|e_u| = e_1$ ($u \in S$) and hence

$$1 = |e| = \sum_{x \in S} |e_x| = ne_1.$$

This yields $e_1 = 1/n$ and the result now follows from Theorem 2.5. $\square$

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