THE ROW RANK OF A SUBRING OF A MATRIX RING

M. S. LI AND J. M. ZELMANOWITZ

ABSTRACT. For R a subring of an $n \times n$ matrix ring $M_n(\Delta)$ over a division ring Δ , we examine an invariant called the row rank of R . Roughly speaking, the row rank of R is the largest integer k such that R contains all k -rowed matrices over a left order in Δ . The row rank of R is then an integer between 0 and n ; and we will see that $\text{row rank } R \geq 1$ means that $M_n(\Delta)$ is the maximal left quotient ring of R , while $\text{row rank } R = n$ signifies that $M_n(\Delta)$ is the classical left quotient ring of R . Thus row rank provides a link between maximal and classical quotient rings for rings of this type. A description of the subrings R with $\text{row rank } R \geq k$ is obtained which subsumes and generalizes earlier theorems of Faith-Utumi and Zelmanowitz, respectively, for the cases $\text{row rank } R = n$ and $\text{row rank } R \geq 1$.

Introduction. Here a subring R of a ring S will be called a *large subring* of S if for every $0 \neq t \in S$ there exists $r \in R$ with $0 \neq rt \in R$. We will be concerned with the situation when $S = M_n(\Delta)$, the ring of $n \times n$ matrices over a division ring Δ . For such subrings the following result was obtained in [4].

THEOREM 1. *The following conditions are equivalent for a subring R of $M_n(\Delta)$, Δ a division ring.*

(1.i) *R is a large subring of $M_n(\Delta)$.*

(1.ii) *There exist matrix units $\{e_{ij} | 1 \leq i, j \leq n\}$ for $M_n(\Delta)$ and a left order D in $\Gamma = \{s \in M_n(\Delta) | se_{ij} = e_{ij}s, 1 \leq i, j \leq n\}$ such that*

$$R \supseteq De_{11} + De_{12} + \cdots + De_{1n} \cong \begin{pmatrix} D & D & \cdots & D \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Recall that $\{e_{ij} | 1 \leq i, j \leq n\}$ is a set of matrix units for $M_n(\Delta)$ if and only if $\sum_{i=1}^n e_{ii} = 1$ and $e_{ij}e_{kl} = \delta_{jk}e_{il}$ for every choice of i, j, k, l ; then the centralizer $\Gamma = \{s \in M_n(\Delta) | se_{ij} = e_{ij}s, 1 \leq i, j \leq n\}$ is a division ring isomorphic to Δ [3, p. 52]. Condition (1.i) says precisely that $M_n(\Delta)$ is the maximal left quotient ring of R [1, p. 66].

The possibility of the preceding theorem was first raised as a question by C. Faith [1, Open Problem 13], and was prompted by the following well-known theorem of Faith and Utumi [2].

THEOREM n . *The following conditions are equivalent for a subring R of $M_n(\Delta)$, Δ a division ring.*

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(n.i) Given any $t \in M_n(\Delta)$ there exists $r \in R$ with r invertible in $M_n(\Delta)$ and with $rt \in R$.

(n.ii) There exist matrix units $\{e_{ij} | 1 \leq i, j \leq n\}$ for $M_n(\Delta)$ and a left order D in $\Gamma = \{s \in M_n(\Delta) | se_{ij} = e_{ij}s, 1 \leq i, j \leq n\}$ such that $R \supseteq \sum_{1 \leq i, j \leq n} De_{ij} \cong M_n(D)$.

The rings in Theorem n are precisely the prime left Goldie orders in $M_n(\Delta)$.

These two theorems suggest the following definition of the (row) rank of a subring R of $M_n(\Delta)$ [5]. For $R \subseteq M_n(\Delta)$ and $0 \leq k \leq n$, we define (row) rank $R \geq k$ if there exist matrix units $\{e_{ij} | 1 \leq i, j \leq n\}$ for $M_n(\Delta)$ and a left order D in $\Gamma = \{s \in M_n(\Delta) | se_{ij} = e_{ij}s, 1 \leq i, j \leq n\}$ such that

$$R \supseteq \sum_{\substack{i=1, \dots, k \\ j=1, \dots, n}} De_{ij} \cong \begin{pmatrix} D & \dots & D \\ \vdots & & \vdots \\ D & \dots & D \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}.$$

Of course we write rank $R = k$ if rank $R \geq k$ but rank $R \not\geq k+1$. In this terminology, conditions (1.ii) and (n.ii) state, respectively, that rank $R \geq 1$ and rank $R = n$. Observe also that the rank of R is defined relative to the specific (inclusion) embedding of R in $M_n(\Delta)$. Later we will see that rank R is essentially independent of the choice of the embedding of R as a large subring of a matrix ring over a division ring (Theorem 2.1).

We let $\Delta^{(n)}$ denote the right Δ -vector space of column n -tuples over Δ . $\Delta^{(n)}$ is a left $M_n(\Delta)$ -module under matrix multiplication, and so is, by restriction, a left R -module. By the rank of an element $r \in R$, written rank r , we mean as usual the dimension of $r\Delta^{(n)}$ as a right Δ -vector space.

In view of Theorem 1 and Theorem n , it was conjectured in [5] that for $R \subseteq M_n(\Delta)$ and $1 \leq k \leq n$, rank $R \geq k$ would be equivalent to the following condition.

(k.i) Given any $0 \neq t \in M_n(\Delta)$ there exists $r \in R$ with rank $r \geq k$ and with $0 \neq rt \in R$.

This conjecture is, however, readily seen to be false. Let Δ be any division ring and set $R = \{(a_{ij}) \in M_3(\Delta) | a_{21} = a_{31} = 0\}$. Then a direct calculation (Example 1.5) establishes that for each $0 \neq t \in M_3(\Delta)$ there exists $r \in R$ with rank $r \geq 2$ and with $0 \neq rt \in R$. On the other hand, as seems intuitively obvious, rank $R = 1$.

Fortunately, the situation can be remedied by strengthening the requirement of condition (k.i). The principal achievement of this article is to establish the following result.

THEOREM k . *The following conditions are equivalent for a subring R of $M_n(\Delta)$, Δ a division ring, $n \geq 2$.*

(k.i*) Given $t_1, \dots, t_{n-1} \in M_n(\Delta)$, not all zero, there exists $r \in R$ with rank $r \geq k$ and with $rt_1, \dots, rt_{n-1} \in R$, not all zero.

(k.ii) rank $R \geq k$.

Theorem 1 and Theorem n are almost immediate consequences of Theorem k in the special cases $k = 1$ and $k = n$. In order to establish Theorem k it will be

necessary to examine carefully the structure of $\Delta^{(n)}$ as a left R -module. The most important techniques in the proof were introduced in [4].

In a sequel to this article we will discuss the relationship between the row rank of R and the dually defined column rank of R .

1. The main theorem. Throughout this article, all rings are associative but need not possess identity elements. For the basic ring theory we need, we rely on [1] as a reference. Theorem k is established by the equivalence of (i) and (ii) in the following theorem.

THEOREM 1.1. *For R a subring of $M_n(\Delta)$, Δ a division ring, $n \geq 2$, and k an integer between 1 and n , the following conditions are equivalent.*

(i) *Given $t_1, \dots, t_{n-1} \in M_n(\Delta)$, not all zero, there exists $r \in R$ with $\text{rank } r \geq k$ and with $rt_1, \dots, rt_{n-1} \in R$, not all zero.*

(ii) $\text{rank } R \geq k$.

(iii) $\Delta^{(n)}$ is a nonsingular indecomposable (quasi-) injective left R -module with $\text{End}_R \Delta^{(n)} \cong \Delta$ acting via right multiplications and $\dim U_\Delta \geq k$ for every nonzero R - Δ -subbimodule U of $\Delta^{(n)}$.

(iv) $M_n(\Delta)$ is the maximal left quotient ring of R and every essential left ideal of R contains an element of rank $\geq k$.

Before turning to the proof of this theorem, we show how to deduce Theorem 1 and Theorem n as special cases. The restriction $n \geq 2$ is a technicality imposed only by the statement of condition (i).

COROLLARY 1.2. *For R a subring of $M_n(\Delta)$, Δ a division ring, the following conditions are equivalent.*

(i) *Given $0 \neq t \in M_n(\Delta)$ there exists $r \in R$ with $0 \neq rt \in R$; i.e., $M_n(\Delta)$ is the maximal left quotient ring of R .*

(ii) $\text{rank } R \geq 1$.

PROOF OF COROLLARY 1.2. (ii) $\Rightarrow$ (i) follows from the corresponding part of Theorem 1.1. If (i) holds, then, as is easily verified, R is a nonsingular large subring of $M_n(\Delta)$. Hence, given any finite sequence $t_1, \dots, t_m \in M_n(\Delta)$, with say $t_1 \neq 0$, $I = \{r \in R \mid rt_i \in R, i = 1, \dots, m\}$ is an essential left ideal of R . Then $It_1 \neq 0$ so that there exists $r \in R$ with $rt_1 \neq 0$ and $rt_i \in R$ for each $i = 1, \dots, m$. This establishes condition (i) of Theorem 1.1 with $k = 1$, and therefore (i) $\Rightarrow$ (ii) in the corollary. $\square$

COROLLARY 1.3. *For R a subring of $M_n(\Delta)$, Δ a division ring, the following conditions are equivalent.*

(i) *Given any $t \in M_n(\Delta)$ there exists $r \in R$ with r invertible in $M_n(\Delta)$ and $rt \in R$; i.e., $M_n(\Delta)$ is the classical left quotient ring of R .*

(ii) $\text{rank } R = n$.

PROOF OF COROLLARY 1.3. Again (ii) $\Rightarrow$ (i) follows from the corresponding part of Theorem 1.1. If we assume (i) of this corollary, then $M_n(\Delta)$ is the classical left quotient ring of R , and so Theorem 1.1(i) with $k = n$ follows from a standard fact about common denominators. Hence $\text{rank } R = n$, completing the proof of this corollary. $\square$

It is also worthwhile noting that a change of matrix units may be necessary before one can detect the rank of a subring. For example, if $R \subseteq M_2(\mathbf{Q})$ is defined by $R = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbf{Z}, a + c = b + d \}$, then R does not contain $\begin{pmatrix} D & D \\ 0 & 0 \end{pmatrix} = \{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} | a, b \in D \}$ for any subdomain D of the rational numbers. If, however, we replace the standard matrix units

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

by the matrix units

$$e_{11} = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}, \quad e_{12} = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}, \quad e_{21} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_{22} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

then $M_2(\mathbf{Q}) = \sum_{i,j=1}^2 \Gamma e_{ij}$, where $\Gamma = \{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} | a \in \mathbf{Q} \} \cong \mathbf{Q}$ and $R = De_{11} + De_{12} + De_{22} \supseteq De_{11} + De_{12}$, where $D = \{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} | a \in \mathbf{Z} \} \cong \mathbf{Z}$. Hence $\text{rank } R \geq 1$ and, in fact, $\text{rank } R = 1$ (see Corollary 1.4(b)).

PROOF OF THEOREM 1.1. We will show the cyclic string of implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i), and also that (iii) $\Leftrightarrow$ (iv). We will frequently quote facts obtained in [4].

(i) $\Rightarrow$ (iii) Condition (i) trivially implies condition (1.i) from which one can readily deduce that $V = \Delta^{(n)}$ is a nonsingular indecomposable injective left R -module with $\text{End}_R V \cong \Delta$ acting via right multiplications [4, Proposition].

Now suppose that U is a nonzero R - Δ -subbimodule of V . Let $0 \neq v \in U$ be arbitrary and complete v to a basis $v, v_1, v_2, \dots, v_{n-1}$ for V over Δ . Choose elements $t_1, \dots, t_{n-1} \in M_n(\Delta)$ satisfying $t_i v = v_i$ for $i = 1, \dots, n-1$. Then apply condition (i) to choose $r \in R$ with $\text{rank } r \geq k$ and with $rt_i \in R$ for each $i = 1, \dots, n-1$. Now $rv \in U$ and $rv_i = r(t_i v) = (rt_i)v \in U$ for each $i = 1, \dots, n-1$. Hence

$$\dim U_\Delta \geq \dim(rv\Delta + rv_1\Delta + \dots + rv_{n-1}\Delta) = \dim rV = \text{rank } r \geq k.$$

(iii) $\Rightarrow$ (ii) This is proved by an argument given in [4]. We will sketch the basic outline and refer the reader to pp. 346–348 of [4] for detailed proofs of the assertions below.

Let $v_1, \dots, v_n$ be any Δ -basis for $\Delta^{(n)}$ and set $A_i = \{r \in R | rv_j = 0 \text{ for each } j = 1, \dots, n, j \neq i\}$, $M = \bigcap_{i=1}^n A_i v_i$, $F = \{\delta \in \Delta | M\delta \subseteq M\}$. Then M is a nonzero compressible left R -module (i.e., M embeds in every nonzero R -submodule of itself) and F is a left order in Δ .

By hypothesis, $h = \dim M\Delta_\Delta \geq k$, so we may choose $m_1, \dots, m_h \in M$ linearly independent over Δ . Complete $m_1, \dots, m_h$ to a Δ -basis $m_1, \dots, m_h, \dots, m_n$ for $\Delta^{(n)}$. Set $N = \bigcap_{i=1}^n C_i m_i$, where $C_i = \{r \in R | rm_j = 0 \text{ for each } j = 1, \dots, n, j \neq i\}$ and put $E = \{\delta \in \Delta | M\delta \subseteq N\}$. Then E is also a left order in Δ .

Next, define $e_{ij} \in M_n(\Delta)$ by the requirements $e_{ij} m_l = \delta_{jl} m_i$ (Kronecker delta) for every choice of $1 \leq i, j, l \leq n$. Then $\{e_{ij} | 1 \leq i, j \leq n\}$ is a set of matrix units for $M_n(\Delta)$ and for each $\alpha \in \Delta$ there exists a unique element $a = a(\alpha) \in \Gamma = \{t \in M_n(\Delta) | te_{ij} = e_{ij}t \text{ for all } 1 \leq i, j \leq n\}$ satisfying $m_i \alpha = a m_i$ for each $1 \leq i \leq n$. Finally, $D = \{a(\alpha) | \alpha \in E\}$ is a left order in Γ and $R \supseteq \sum_{1 \leq i \leq h, 1 \leq j \leq n} De_{ij}$. Hence $\text{rank } R \geq h \geq k$. (Observe, for later use, that we have actually shown that $\text{rank } R \geq \dim M\Delta_\Delta$.)

(ii) $\Rightarrow$ (i) We will show a bit more than necessary. Let $t_1, \dots, t_m \in M_n(\Delta)$ be any finite sequence with $t_1 \neq 0$; we claim that there then exists $r \in R$ with $\text{rank } r \geq k$, $rt_1 \neq 0$, and $rt_i \in R$ for all $i = 1, \dots, m$.

We are given that $M_n(\Delta) = \sum_{i,j=1}^n \Gamma e_{ij}$ and $R \supseteq \sum_{1 \leq i \leq k; 1 \leq j \leq n} \Delta e_{ij}$, where Δ is a left order in $\Gamma = \{t \in M_n(\Delta) | te_{ij} = e_{ij}t \text{ for each } 1 \leq i, j \leq n\}$. Hence we may write each $t_h = \sum_{i,j=1}^n a^{-1} b_{hij} e_{ij}$ with $a, b_{hij} \in D$ for $1 \leq i, j \leq n$ and $1 \leq h \leq m$. Also $b_{1i_0j_0} \neq 0$ for some choice of $1 \leq i_0, j_0 \leq n$ because $t_1 \neq 0$. Define $r \in R$ by

$$r = \begin{cases} a(e_{11} + \dots + e_{k-1,k-1} + e_{kk}) & \text{if } i_0 \leq k, \\ a(e_{11} + \dots + e_{k-1,k-1} + e_{ki_0}) & \text{if } i_0 > k. \end{cases}$$

It is then routine to check that for each $h = 1, \dots, m$, $rt_h \in \sum_{1 \leq i \leq k; 1 \leq j \leq n} \Delta e_{ij} \subseteq R$, that $rt_1 \neq 0$, and that $\text{rank } r = \dim r\Delta_{\Delta}^{(n)} = k$. We omit the details.

(iii) $\Rightarrow$ (iv) Since we have already shown that (iii) $\Leftrightarrow$ (i) it is evident that $M_n(\Delta)$ is the maximal left quotient ring of R . It remains to show that if L is an essential left ideal of R , then L contains an element of rank $\geq k$.

Choose a Δ -basis $v_1, \dots, v_n$ for $\Delta^{(n)}$ and set $A_i = \{r \in R | rv_j = 0 \text{ for } j = 1, \dots, n, j \neq i\}$. $A_i \cong A_i v_i$ under the correspondence $a \leftrightarrow av_i$ for $a \in A_i$ because $\Delta^{(n)}$ is a faithful left R -module. Since L is an essential left ideal of R and $\Delta^{(n)}$ is a uniform left R -module, $\bigcap_{i=1}^n (L \cap A_i) v_i \neq 0$. By hypothesis there exist elements $u_1, \dots, u_k \in \bigcap_{i=1}^n (L \cap A_i) v_i$, linearly independent over Δ . For each $j = 1, \dots, k$ we may write $u_j = a_j v_j$ with $a_j \in L \cap A_j$. Set $a = \sum_{j=1}^k a_j \in L$. Since $u_j = av_j$ it is clear that $\text{rank } a \geq k$, and so a is the desired element of L .

(iv) $\Rightarrow$ (iii) Clearly (iv) implies (1.i) and so, as in the proof of (i) $\Rightarrow$ (iii), we can conclude that $\Delta^{(n)}$ is a nonsingular indecomposable injective left R -module with $\text{End}_R \Delta^{(n)} \cong \Delta$ acting via right multiplications.

Now let U be a nonzero R - Δ -subbimodule of $\Delta^{(n)}$ and fix a Δ -basis $v_1, \dots, v_n$ for $\Delta^{(n)}$. Then, because $\Delta^{(n)}$ is a uniform R -module, $L = \{r \in R | rv_i \in U, 1 \leq i \leq n\}$ is an essential left ideal of R . By hypothesis, there exists $r \in L$ with $\text{rank } r = \dim r\Delta_{\Delta}^{(n)} \geq k$. Since $r\Delta^{(n)} = \sum_{i=1}^n rv_i \Delta \subseteq U\Delta \subseteq U$, $\dim U_{\Delta} \geq k$.

With this, the proof is completed. $\square$

In the course of proving Theorem 1.1 we have obtained some additional information which we now elaborate on and display.

COROLLARY 1.4. *Suppose that R is a subring of $M_n(\Delta)$ with $\text{rank } R = k \geq 1$. Then the following are true.*

(a) *Given any elements $t_1, \dots, t_m$ of $M_n(\Delta)$ with $t_1 \neq 0$ there exists $r \in R$ with $\text{rank } r \geq k$, $rt_1 \neq 0$ and $rt_i \in R$ for each $i = 1, \dots, m$.*

(b) *For any Δ -basis $v_1, \dots, v_n$ of $\Delta^{(n)}$, $\bigcap_{i=1}^n A_i v_i$ is a nonzero compressible left R -module and $\text{rank } R = \dim(\bigcap_{i=1}^n A_i v_i)\Delta_{\Delta}$, where $A_i = \{r \in R | rv_j = 0 \text{ for } 1 \leq j \leq n, j \neq i\}$.*

(c) *$\Delta^{(n)}$ contains a unique minimal R - Δ -subbimodule U with $\dim U_{\Delta} = \text{rank } R$.*

PROOF. (a) was proven in showing that (ii) $\Rightarrow$ (i) in Theorem 1.1, and the first assertion in (b) was noted in (iii) $\Rightarrow$ (ii). It is also clear from Theorem 1.1 that when $\text{rank } R \geq 1$, $\text{rank } R = \min\{\dim U_{\Delta} | U \text{ is a nonzero } R\text{-}\Delta\text{-subbimodule of } \Delta^{(n)}\}$. Hence the proof of this corollary can be completed by showing that given any Δ -basis $v_1, \dots, v_n$ of $\Delta^{(n)}$, $(\bigcap_{i=1}^n A_i v_i)\Delta$ is the unique minimal nonzero R - Δ -subbimodule of $\Delta^{(n)}$, where $A_i = \{r \in R | rv_j = 0 \text{ for } 1 \leq j \leq n, j \neq i\}$.

To see this, set $M = \bigcap_{i=1}^n A_i v_i$. If now W is any nonzero R - Δ -subbimodule of $\Delta^{(n)}$ then $M \cap W \neq 0$ because $\Delta^{(n)}$ is uniform. Since M is compressible there exists an R -monomorphism $\alpha: M \rightarrow M \cap W$, and, because $\Delta^{(n)}$ is quasi-injective, α extends to an element of $\text{End}_R \Delta^{(n)} \cong \Delta$ which we also denote by α . Hence $M\alpha \subseteq M \cap W \subseteq W$ and therefore $M\Delta = M\alpha\Delta \subseteq W\Delta \subseteq W$, which proves that $M\Delta$ is the unique minimal R - Δ -subbimodule of $\Delta^{(n)}$. From part (b) we know that $\dim M\Delta_\Delta = \text{rank } R$. $\square$

EXAMPLE 1.5. Let Δ be a field and set $R = \{(a_{ij}) \in M_3(\Delta) | a_{21} = a_{31} = 0\}$, a subring of $M_3(\Delta)$. We verify here the two assertions made in the Introduction.

(a) For each $0 \neq t \in M_3(\Delta)$ there exists $r \in R$ with $\text{rank } r \geq 2$ and with $0 \neq rt \in R$; and

(b) $\text{rank } R = 1$.

For (a), let $0 \neq t = (a_{ij}) \in M_3(\Delta)$ be given. If $a_{21} = a_{31} = 0$, then $t \in R$ and we may choose r to be the identity matrix. If $a_{21}a_{31} \neq 0$, then choose

$$r = \begin{pmatrix} 0 & 0 & 1 \\ 0 & a_{31} & -a_{21} \\ 0 & 0 & 0 \end{pmatrix}.$$

If $a_{21} \neq 0$ and $a_{31} = 0$, then choose

$$r = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix};$$

similarly, if $a_{21} = 0$ and $a_{31} \neq 0$ take

$$r = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In every case, $\text{rank } r \geq 2$ and $0 \neq rt \in R$.

For (b), we apply Corollary 1.4(b) with

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Then

$$\bigcap_{i=1}^3 A_i v_i = \begin{pmatrix} \Delta \\ 0 \\ 0 \end{pmatrix}$$

and so $\text{rank } r = \dim(\bigcap_{i=1}^3 A_i v_i)\Delta_\Delta = 1$.

2. Independence of rank. The definition of rank R that we used in §1 was dependent on the (inclusion) embedding of R in $M_n(\Delta)$. The purpose of this section is to show that, when it is ≥ 1 , the rank of R is fundamentally independent of the choice of the embedding of R in a matrix ring over a division ring. We begin by slightly extending the definition of the rank of R .

Let $\phi: R \rightarrow M_n(\Delta)$, Δ a division ring, be a ring monomorphism such that, regarding $\phi(R)$ as a subring of $M_n(\Delta)$, $\text{rank } \phi(R) = k$. Then we write $\text{rank}_\phi R = k$.

THEOREM 2.1 (*Independence of rank*). *Let $\phi: R \rightarrow M_n(\Delta)$ and $\psi: R \rightarrow M_m(\Gamma)$ be ring monomorphisms with Δ and Γ division rings. If $\text{rank}_\phi R \geq 1$ and $\text{rank}_\psi R \geq 1$, then $n = m$, $\Delta \cong \Gamma$, and $\text{rank}_\phi R = \text{rank}_\psi R$.*

It should be evident that the case when $\text{rank}_\phi R = 0$ or $\text{rank}_\psi R = 0$ must be excluded from consideration. For example, if $\phi: \Delta \rightarrow M_1(\Delta) = \Delta$ is the identity map and $\psi: \Delta \rightarrow M_2(\Delta)$ is defined by $\psi(x) = \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$, then $\text{rank}_\phi \Delta = 1$ while $\text{rank}_\psi \Delta = 0$.

From part (iii) of Theorem 1.1 and Corollary 1.4(c) we know that if $\text{rank}_\phi R \geq 1$, then (with respect to the R -module structure naturally induced by $\phi: R \rightarrow M_n(\Delta)$) $\Delta^{(n)}$ is a faithful nonsingular indecomposable injective left R -module with $\text{End}_R \Delta^{(n)} \cong \Delta$ (acting as right multiplications) and with $\text{rank}_\phi R = \dim U_\Delta$, where U is the unique minimal R - Δ -subbimodule of $\Delta^{(n)}$. A similar statement is also true for $\Gamma^{(m)}$ as a left R -module. Hence the preceding theorem is established by the following standard proposition. We include a proof for the sake of completeness.

PROPOSITION 2.2. *If V is a faithful nonsingular indecomposable injective left R -module and W is any nonsingular indecomposable injective left R -module, then $W \cong V$.*

PROOF. Choose $0 \neq w \in W$. Since W is nonsingular and indecomposable there exists a uniform left ideal I of R with $Iw \cong I$ under the R -isomorphism ϕ defined by $\phi(rw) = r$ for $r \in I$. Since V is faithful there exists $v \in V$ with $Iv \neq 0$; then $\psi(r) = rv$ for $r \in I$ defines an R -isomorphism from I onto Iv because I is uniform and V is nonsingular. Since V is injective and W is indecomposable, $\phi\psi$ extends to an R -monomorphism θ of W into V . θ is the desired isomorphism because W is injective and V is indecomposable. $\square$

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA BARBARA, CALIFORNIA 93106