ON CROSSED PRODUCTS WITH PROPERTY T

SHŌICHI WATANABE

ABSTRACT. Let N be a finite von Neumann algebra (with faithful normal normalized trace τ), G a countable discrete group, and α a τ -preserving action of G on N such that $N \rtimes_{\alpha} G$ is a factor. It is proved that if $N \rtimes_{\alpha} G$ has Property T, then G has Kazhdan's Property T.

Connes [3] defined Property T for type II_1 -factors and claimed that a discrete group G has Kazhdan's Property T if and only if its von Neumann algebra has this property. In [4] Connes and Jones extended this property to any von Neumann algebra which was also called Property T. In this paper we discuss Property T for crossed products.

Here we recall an equivalent condition for countable discrete groups to have Kazhdan's Property T. A countable discrete group G has Property T if and only if there exist a finite subset F of G and $\varepsilon > 0$ such that

- (*) any unitary representation $\{\omega, H_{\omega}\}$ of G with unit vector $\eta \in H_{\omega}$ such that $\|\omega(g)\eta - \eta\| < \varepsilon$ for all $g \in F$ has a nonzero fixed vector.

Let N be a von Neumann algebra. We say that a Hilbert space H is a correspondence from N to N if there exist two normal commuting representations of N , one of which is an antirepresentation, on H . If we let ρ_l, ρ_r be such representations (ρ_r is anti) on H , then we can write

$$x\xi y = \rho_l(x)\rho_r(y)\xi, \quad x, y \in N, \xi \in H.$$

Of course, the standard representation of N becomes a correspondence from N to N automatically. According to [4], a II_1 -factor N has Property T if and only if there exist $x_1, \dots, x_n \in N$, $\varepsilon > 0$, and $K > 0$ such that

- (**) for any $\delta \leq \varepsilon$ and correspondence H from N to N with vector $\eta \in H$, $\|\eta\| = 1$, $\|x_i\eta - \eta x_i\| < \delta$, there exists a central vector $\xi \in H$ (i.e., $x\xi = \xi x$ for all $x \in N$) such that $\|\xi - \eta\| < K\delta$.

Let N act on a Hilbert space K and α be an action of G on N (i.e., α is a homomorphism from G into the $*$ -automorphism group of N). Then the crossed product $N \rtimes_{\alpha} G$ is a von Neumann algebra on $l^2(G, K)$ generated by $\{\pi(a); a \in N\}$ and $\{v(g); g \in G\}$, where for $\xi \in l^2(G, K)$, $g, h \in G$, and $a \in N$,

$$(\pi(a)\xi)(g) = \alpha_g^{-1}(a)\xi(g), \quad (v(g)\xi)(h) = \xi(g^{-1}h).$$

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To each $x \in N \rtimes_{\alpha} G$ one can associate a function $x(g): G \rightarrow N$ and x is represented by $\sum_{g \in G} \pi(x(g))v(g)$ (Fourier expansion). If N has a faithful normal normalized trace τ and α is τ -preserving, then $\tilde{\tau}(x) = \tau(x(e))$ defines a faithful normal normalized trace of $N \rtimes_{\alpha} G$, where e is the identity of G (cf. [5]).

Now we consider the case when N is a II_1 -factor and α is an action of G on N such that $N \rtimes_{\alpha} G$ is a II_1 -factor. In this case, we can verify that if N and G have Property T, then $N \rtimes_{\alpha} G$ has Property T. In fact let $x_1, \dots, x_n \in N$, $\varepsilon_1 > 0$, $K > 0$, $F \subseteq G$, and $\varepsilon_2 > 0$ satisfy the condition of (*) and (**) respectively, and H be any correspondence from N to N with $\eta \in H$, $\|\eta\| = 1$, $\|\pi(x_i)\eta - \eta\pi(x_i)\| < \delta$, and $\|v(g)\eta - \eta v(g)\| < \delta$ ($i = 1, \dots, n$, $g \in F$) for δ small enough. Then since N has Property T, we can find a vector ξ near η which is central for $\pi(N)$. Let H_1 be the closed linear span of $\{v(g)\xi v(g)^*; g \in G\}$. Then for any $\zeta \in H_1$, $\zeta \rightarrow v(g)\zeta v(g)^*$ induces a unitary representation of G on H_1 which has nearly fixed vector ξ . Since G has Property T, we can find a nonzero fixed vector in H_1 and it becomes a central vector for $N \rtimes_{\alpha} G$, so that $N \rtimes_{\alpha} G$ has Property T.

Conversely, suppose that $N \rtimes_{\alpha} G$ has Property T. In this case, N does not necessarily have Property T. A counterexample is shown by M. Choda [1]. But the next theorem says that the group G needs Property T.

THEOREM. *Let N be a finite von Neumann algebra with faithful normal normalized trace τ , G a countable discrete group, and α a τ -preserving action of G on N such that $N \rtimes_{\alpha} G$ is a factor with Property T. Then G has Kazhdan's Property T.*

PROOF. By our assumption, $N \rtimes_{\alpha} G$ is a II_1 -factor with faithful normal normalized trace $\tilde{\tau}(x) = \tau(x(e))$ ($x \in N \rtimes_{\alpha} G$). Let N act on $L^2(N, \tau)$ standardly, J be a canonical involution on $L^2(N, \tau)$, and $\xi_0 \in L^2(N, \tau)$ an image of the identity in N . For $g \in G$ let u_g be the unitary on $L^2(N, \tau)$ defined by $u_g x \xi_0 = \alpha_g(x) \xi_0$ ($x \in N$). Then $\{u_g\}_{g \in G}$ is a canonical unitary implementation of α . Since $N \rtimes_{\alpha} G$ has Property T, there exists $x_1, \dots, x_n \in N$, $\varepsilon > 0$, and $K > 0$ which satisfy (**). Let $\sum_{g \in G} \pi(x_i(g))v(g)$ be the Fourier expansion of x_i for $i = 1, \dots, n$. Then for $\delta \leq \min\{\varepsilon, 1/K\}$ there exists a finite subset F of G such that

$$\left\| \sum_{g \in G \setminus F} \pi(x_i(g))v(g) \right\|_2 < \delta/3 \quad \text{for all } i = 1, \dots, n,$$

where $\|x\|_2 = \tilde{\tau}(x^*x)^{1/2}$. Let $\{\omega, H_{\omega}\}$ be a unitary representation of G with $\eta \in H_{\omega}$, $\|\eta\| = 1$, and

$$\|\omega(f)\eta - \eta\| < \delta/3|F| \max\{\|x_i(g)\|_2; i = 1, \dots, n, g \in F\}$$

(for all $f \in F$), where $|F|$ is the number of the elements in F . We define two representations $\{\pi_l, v_l\}, \{\pi_r, v_r\}$ of G and N on $H = l^2(G, L^2(N, \tau) \otimes H_{\omega})$ as follows,

$$\begin{aligned} (\pi_l(x)\xi)(g) &= (\alpha_g^{-1}(x) \otimes 1)\xi(g), & (v_l(g)\xi)(h) &= \xi(g^{-1}h), \\ (\pi_r(x)\xi)(g) &= (Jx^*J \otimes 1)\xi(g), & (v_r(g)\xi)(h) &= (u_g \otimes \omega(g))^*\xi(hg^{-1}). \end{aligned}$$

Then $\{\pi_l, v_l\}$ and $\{\pi_r, v_r\}$ induce a normal representation and an antirepresentation of $N \rtimes_{\alpha} G$ respectively and they commute with each other. Let ζ be the vector in

H defined by

$$\zeta(g) = \begin{cases} \xi_0 \otimes \eta & \text{if } g = e, \\ 0 & \text{otherwise.} \end{cases}$$

Then we can easily see that ζ is a trace vector (i.e., $(x\zeta|\zeta) = (\zeta x|\zeta) = \tilde{\tau}(x)$ for $x \in N \rtimes_\alpha G$). Thus we have

$$\begin{aligned} \|x_i\zeta - \zeta x_i\| &\leq \left\| \sum_{g \in G \setminus F} \pi_l(x_i(g))v_l(g)\zeta \right\| \\ &\quad + \sum_{g \in F} \|\pi_l(x_i(g))v_l(g)\zeta - v_r(g)\pi_r(x_i(g))\zeta\| + \left\| \sum_{g \in G \setminus F} v_r(g)\pi_r(x_i(g))\zeta \right\| \\ &\leq 2\delta/3 + \sum_{g \in F} \|\pi_l(x_i(g))v_l(g)\zeta - v_r(g)\pi_r(x_i(g))\zeta\| \\ &= 2\delta/3 + \sum_{g \in F} \|(\alpha_g^{-1}(x_i(g)) \otimes 1)\xi_0 \otimes \eta - (u_g^*x_i(g) \otimes \omega(g)^*)\xi_0 \otimes \eta\| \\ &= 2\delta/3 + \sum_{g \in F} \|\alpha_g^{-1}(x_i(g))\xi_0\| \|\omega(g)\eta - \eta\| \quad (\text{since } u_g\xi_0 = \xi_0) \\ &< \delta. \end{aligned}$$

Thus there exists a central vector $\tilde{\zeta}$ in H for the crossed product such that $\|\tilde{\zeta} - \zeta\| < K\delta \leq 1$. Let $\{\varepsilon_i\}_{i=0}^\infty$ be an orthonormal base of $L^2(N, \tau)$ such that $\varepsilon_0 = \xi_0$. If we write $\tilde{\zeta}(e) = \sum_{i=0}^\infty \varepsilon_i \otimes \zeta_i$, then

$$\|\zeta_0 - \eta\| \leq \|\tilde{\zeta} - \zeta\| < 1.$$

Thus ζ_0 is a nonzero vector in H . Since $\tilde{\zeta}(e) = (u_g \otimes \omega(g))\tilde{\zeta}(e)$, we have

$$\varepsilon_0 \otimes \zeta_0 + \sum_{i=1}^\infty \varepsilon_i \otimes \zeta_i = \varepsilon_0 \otimes \omega(g)\zeta_0 + \sum_{i=1}^\infty u_g \varepsilon_i \otimes \omega(g)\zeta_i$$

for all $g \in G$. Hence $\varepsilon_0 \otimes \zeta_0 = \varepsilon_0 \otimes \omega(g)\zeta_0$ because $\{u_g \varepsilon_i\}_{i=1}^\infty$ are orthogonal to ε_0 , so that $\omega(g)\zeta_0 = \zeta_0$ for all $g \in G$. Thus G has Property T.

As in the case of crossed products, it arises naturally to ask the condition for tensor products to have Property T. We know that if M and N are II_1 -factors with Property T, then $M \overline{\otimes} N$ has Property T by Connes' note. However it is unknown whether M and N (M or N) have Property T or not when $M \overline{\otimes} N$ has Property T as far as we know.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, YAMAGATA UNIVERSITY,
YAMAGATA, 990, JAPAN