MOST NORMAL OPERATORS ARE DIAGONAL

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ABSTRACT. We show that nondiagonal operators form a meager set in the set of all normal operators on a separable, infinite dimensional Hilbert space.

Let $\mathcal{H}$ be a separable, infinite dimensional complex Hilbert space. We denote by $\mathcal{N}$ the set of all (bounded) normal operators on $\mathcal{H}$. Notice that $\mathcal{N}$ is closed (in the norm topology). An operator D on $\mathcal{H}$ is said to be diagonal if there is an orthonormal basis $\{e_n\}$ and a sequence of complex numbers $\{\lambda_n\}$ such that $De_n = \lambda_n e_n$ for all n . Thus, the diagonal operators are those in $\mathcal{N}$ whose eigenvectors span the whole space.

It follows from the spectral theory that every normal operator can be approximated (in norm) by diagonal operators. Even so, diagonal operators are usually considered to be rather special and thus one may think that such operators are rare in some sense. However, the following result shows the opposite: most normal operators are diagonal (in the sense of Baire's categories).

THEOREM. *The set $\mathcal{D}$ of diagonal operators contains the intersection of a countable family of open dense subsets of $\mathcal{N}$.*

PROOF. Let $\{P_n\}$ be an increasing sequence of (orthogonal) projections with finite ranks such that $\|P_n x - x\| \rightarrow 0$ as $n \rightarrow \infty$ for all x in $\mathcal{H}$. Let $\mathcal{N}_n$ be the set of all those normal operators N each of which satisfies the following condition: There is a finite rank projection E (depending on N) such that

- (i) $EN = NE$,
- (ii) $\|P_n EP_n - P_n\| < n^{-1}$,
- (iii) $\sigma(N|E\mathcal{H})$ and $\sigma(N|(1-E)\mathcal{H})$ are disjoint.

Here, $\sigma(N|E\mathcal{H})$ stands for the spectrum of $N|E\mathcal{H}$, the restriction of N to the range of E . To see that each $\mathcal{N}_n$ is a dense open subset of $\mathcal{N}$, notice that, by means of Dunford's functional calculus, the condition (iii) guarantees that any normal operator close enough to N has a finite rank spectral projection close to E . It remains to show that $\bigcap_{n=1}^{\infty} \mathcal{N}_n \subseteq \mathcal{D}$.

Suppose that $N \in \bigcap_{n=1}^{\infty} \mathcal{N}_n$. Let F be the projection of $\mathcal{H}$ onto the subspace spanned by eigenvectors of N . It suffices to show that $F = I$. Notice that F commutes with every operator that commutes with N . For each n , our assumption $N \in \mathcal{N}_n$ implies the existence of a finite rank projection E such that (i)–(iii) above are satisfied. From (i) we get $E \leq F$. By (ii), we have

$$P_n F P_n \geq P_n E P_n \geq P_n - n^{-1} I.$$

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Now let $n \rightarrow \infty$. Since $P_n \uparrow I$ in the strong operator topology, we have $F \geq I$. Therefore $F = I$. The proof is complete.

ADDED IN PROOF. Recently the author has received a preprint from D. A. Herero entitled *Most quasitriangular operators are triangular, most biquasitriangular operators are bitriangular*.

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