ON A MINIMAX INEQUALITY OF KY FAN

CHUNG-WEI HA

Dedicated to Professor Ky Fan on his 70th birthday

ABSTRACT. An extension of a minimax inequality of Ky Fan is given. It is then used to generalize to the set-valued setting a fixed point theorem of Ky Fan.

Since Fan gave his minimax inequality in [4], various extensions of this interesting result have been obtained (see [5] and the references there). One was proved in [6, Theorem 3], which can be formulated in the following form:

THEOREM 1. *Let E, F be Hausdorff topological vector spaces, $X \subset E$, $Y \subset F$ be nonempty convex subsets, Y be compact. Let f be a real-valued function defined on $X \times Y$ such that*

- (a) *for each $y \in Y$, $f(x, y)$ is quasiconvex in $x \in X$;*
- (b) *for each $x \in X$, $f(x, y)$ is upper semicontinuous in $y \in Y$.*

If T is a upper semicontinuous set-valued map defined on X such that Tx is a nonempty closed convex subset of Y for each $x \in X$, then

$$\inf_{y \in Tx} f(x, y) \leq \max_{y \in Y} \inf_{x \in X} f(x, y).$$

When $X = Y$ and the set-valued map T is defined by $Tx = \{x\}$ for $x \in X$, Theorem 1 reduces to Fan's minimax inequality [4, Theorem 1]. We recall that a real-valued function ϕ defined on a convex set X is called *quasiconvex* if, for each real number t , the set $\{x \in X: \phi(x) < t\}$ is convex. Let X, Y be topological spaces. By a *set-valued map* T defined on X with values in Y , we mean a map T that assigns to each $x \in X$ a unique nonempty subset Tx of Y . T is called *upper semicontinuous* on X if, for each open subset G of Y , the set $\{x \in X: Tx \subset G\}$ is open in X . It is easy to show (e.g. [1, p. 72]) that if Y is compact Hausdorff and if Tx is closed for each $x \in X$, then T is upper semicontinuous on X if and only if the graph $\{(x, y) \in X \times Y: y \in Tx\}$ of T is closed in $X \times Y$.

In [4] Fan also gave a version of the minimax inequality with a lopsided saddle point (see [4, Corollary 1]) which is useful in many situations. The main result of this note is the following Theorem 2, which is a variant of Theorem 1 with a lopsided saddle point.

THEOREM 2. *Let E, F be Hausdorff topological vector spaces, $X \subset E$, $Y \subset F$ be nonempty compact convex sets. If g is a continuous real-valued function defined on $X \times Y$, and if T is an upper semicontinuous set-valued map defined on X such that*

- (c) *for each $y \in Y$, $g(x, y)$ is quasiconvex in $x \in X$;*

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(d) for each $x \in X$, Tx is a nonempty closed convex subset of Y , then there exist $x_0 \in X$ and $y_0 \in Tx_0$ satisfying

$$(1) \quad g(x_0, y_0) \leq g(x, y_0)$$

for all $x \in X$.

PROOF. We define a real-valued function on $X \times Y$ by

$$f(x, y) = g(x, y) - \min_{z \in X} g(z, y).$$

Clearly f satisfies condition (a) in Theorem 1. Moreover, it is easy to show (e.g. [1, p. 70]) that f is continuous on $X \times Y$. By the compactness of X , for each $y \in Y$ there exists $x \in X$ such that $f(x, y) = 0$ and hence, by Theorem 1,

$$(2) \quad \inf_{y \in Tx} f(x, y) \leq 0.$$

Since the graph of T is closed and so is compact in $X \times Y$, the infimum in (2) is attained. The result now follows.

As an application, we use Theorem 2 to prove the following Theorem 3, which is a generalization to the set-valued setting of Fan's fixed point theorem [3, Theorem 1].

THEOREM 3. *Let E be a locally convex Hausdorff topological vector space, $X \subset E$ be a nonempty compact convex subset. Let T be an upper semicontinuous set-valued map defined on X such that Tx is a compact convex subset of E for each $x \in X$. Then either T has a fixed point in X , or there exist $x_0 \in X$, $u_0 \in Tx_0$, and a continuous seminorm p on E such that*

$$(3) \quad 0 < p(x_0 - u_0) \leq p(x - u_0)$$

for all $x \in X$.

PROOF. Suppose that T has no fixed point in X . Then for each $x \in X$, the origin 0 of E does not belong to the compact set $x - Tx$ and so there exist $\delta_x > 0$ and a continuous seminorm p_x on E such that $p_x(x - u) > 2\delta_x$ for all $u \in Tx$. By the upper semicontinuity of T , there exists an open neighborhood N_x of x in X such that $p_x(z - v) > \delta_x$ for all $z \in N_x$ and $v \in Tz$. Since the family $\{N_x : x \in X\}$ is an open cover of the compact set X , there exists a finite subset $\{x_1, \dots, x_m\}$ of X such that $\{N_{x_i} : 1 \leq i \leq m\}$ covers X . Let $p = \max\{p_{x_i} : 1 \leq i \leq m\}$ and $\delta = \min\{\delta_{x_i} : 1 \leq i \leq m\} > 0$. Then p is a continuous seminorm on E and

$$(4) \quad p(x - u) > \delta$$

for all $x \in X$ and $u \in Tx$.

It can be shown (e.g. [1, p. 72]) that the image $T(X) = \bigcup\{Tx \subset Y : x \in X\}$ of T is compact. As the assumptions on X and T remain unchanged in the completion of E , without loss of generality we can assume that E is complete. Let Y be the closed convex hull of the image $T(X)$ of T in E , then Y is compact convex. We define a real-valued function g on $X \times Y$ by $g(x, y) = p(x - y)$. Clearly g and T satisfy the hypothesis of Theorem 2. (3) now follows from (1) and (4).

As a direct consequence of Theorem 3, we have the following

THEOREM 4. *Let E be a locally convex Hausdorff topological vector space, $X \subset E$ be a nonempty compact convex subset. Let T be an upper semicontinuous set-valued map defined on X such that Tx is a compact convex subset of E for each $x \in X$. If T satisfies the following condition:*

(e) *for any $x \in X$, $u \in Tx$, there exists a number λ (real or complex, depending on whether the vector space E is real or complex) such that*

$$|\lambda| < 1 \quad \text{and} \quad \lambda x + (1 - \lambda)u \in X,$$

then T has a fixed point in X .

PROOF. Suppose that T has no fixed point in X . By Theorem 3, there exist $x_0 \in X$, $u_0 \in Tx_0$ and a continuous seminorm p on E such that (3) holds. For these x_0 and $u_0 \in Tx_0$, by (e) there exists a number λ such that $|\lambda| < 1$ and $x = \lambda x_0 + (1 - \lambda)u_0 \in X$. Then

$$0 < p(x_0 - u_0) \leq |\lambda|p(x_0 - u_0),$$

which contradicts $|\lambda| < 1$.

Theorem 4 generalizes to the set-valued setting Fan's result [3, Theorem 3]. It is new only when the vector space E is complex and the number λ in the condition (e) is allowed to be complex. If the vector space E is real, more general results have been obtained (see [4, 5]). When the image $T(X)$ of T is contained in X , the condition (e) is trivially satisfied and Theorem 4 reduces to the well-known fixed point theorem of Fan [2, Theorem 1]. An extension of [3, Theorem 3] was proved under a stronger hypothesis in [7], which does not contain Fan's fixed point theorem [2, Theorem 1].

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DEPARTMENT OF MATHEMATICS, NATIONAL TSING HUA UNIVERSITY, HSINCHU, TAIWAN, R.O.C.