ON SOME SPECTRAL PROPERTIES OF ELLIPTIC PSEUDODIFFERENTIAL OPERATORS

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ABSTRACT. We prove that the minimal and maximal operators associated with an elliptic pseudodifferential operator coincide in $L^p(\mathbf{R}^n)$, $1 < p < \infty$. We obtain a set of necessary and sufficient conditions for a measurable function q on $\mathbf{R}^n$ to be compact relative to some integral power of a constant coefficient elliptic pseudodifferential operator.

1. Introduction. For $m \in (-\infty, \infty)$, we denote by S^m the set of all functions σ in $C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ such that, for all multi-indices α and β , there exists a constant $C_{\alpha\beta} > 0$ satisfying

$$|(D_x^\alpha D_\xi^\beta \sigma)(x, \xi)| \leq C_{\alpha\beta}(1 + |\xi|)^{m-|\beta|}$$

for all x, ξ in $\mathbf{R}^n$. Let $\sigma \in S^m$. Then we define the operator T_σ on $\mathcal{S}$ (the Schwartz space) by

$$(T_\sigma \phi)(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{ix \cdot \xi} \sigma(x, \xi) \hat{\phi}(\xi) d\xi$$

for all ϕ in $\mathcal{S}$, where

$$\hat{\phi}(\xi) = \int_{\mathbf{R}^n} e^{-ix \cdot \xi} \phi(x) dx.$$

We call T_σ a pseudodifferential operator with symbol σ in S^m . The smallest number m for which $\sigma \in S^m$ is called the order of σ (or T_σ). Furthermore, we call σ an elliptic symbol or T_σ an elliptic pseudodifferential operator if there exist positive constants C and R such that

$$|\sigma(x, \xi)| \geq C(1 + |\xi|)^m$$

for $|\xi| \geq R$.

The aim of this paper is to study some spectral properties of elliptic pseudodifferential operators on $L^p(\mathbf{R}^n)$, $1 < p < \infty$. In §2, we recall some standard results of pseudodifferential operators and derive some easy consequences which we shall use afterwards. In §3, we prove that the minimal and maximal operators associated with an elliptic pseudodifferential operator with symbol in S^m coincide in $L^p(\mathbf{R}^n)$, $1 < p < \infty$. This result can be generalized to a wider class of pseudodifferential operators with symbols in $S_{\rho, \delta}^m$. See the monograph [3] by Kumano-go for a comprehensive account of these operators. If $\sigma \in S^m$ is any symbol independent of x , then it follows from Theorem 2.4 in Wong [7] that the minimal and maximal operators associated with T_σ coincide in $L^p(\mathbf{R}^n)$, $1 \leq p < \infty$. In §4 and 5,

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we study operators of the form $T_\sigma^k + q$, where σ is an elliptic symbol of positive order m , T_σ^k the k th power of T_σ and q a measurable function on $\mathbf{R}^n$. Such operators come up naturally in quantum physics. The most celebrated one has symbol $\tau(\xi) = \sqrt{|\xi|^2 + \mu^2}$. The operator $T_\tau + q$ is reminiscent of the relativistic Hamiltonian of a spinless particle of mass μ under the influence of the potential q . A main result in the paper is a characterization of functions q which are compact relative to some power of T_σ , where σ is an elliptic symbol of positive order and independent of x in $\mathbf{R}^n$. The special case when T_σ is a constant coefficient ordinary differential operator can be found in §9 of Chapter 14 in Schechter [6]. For the connection between relative compactness and the invariance of the essential spectrum, see Gustafson and Weidmann [2], and Schechter [4].

Finally, it should be pointed out that Theorems 4.1 and 5.1 in this paper are in fact equivalent to the corresponding results obtained by Balslev in [1] for elliptic partial differential operators with constant coefficients. However, an approach via pseudodifferential operators is perhaps also enlightening.

2. Some fundamental results. We recall some fundamental properties of pseudodifferential operators.

THEOREM 2.1. *Let $\sigma_1 \in S^{m_1}$ and $\sigma_2 \in S^{m_2}$. Then the composition $T_{\sigma_1}T_{\sigma_2}$ is a pseudodifferential operator with symbol σ in $S^{m_1+m_2}$. Moreover, $\sigma(x, \xi)$ has the asymptotic expansion*

$$(2.1) \quad \sigma(x, \xi) \sim \sum_{\alpha} \frac{(-i)^{|\alpha|}}{\alpha!} [\partial_{\xi}^{\alpha} \sigma_1](x, \xi) [\partial_x^{\alpha} \sigma_2](x, \xi).$$

Here, formula (2.1) means that

$$\sigma(x, \xi) - \sum_{|\alpha| < N} \frac{(-i)^{|\alpha|}}{\alpha!} [\partial_{\xi}^{\alpha} \sigma_1](x, \xi) [\partial_x^{\alpha} \sigma_2](x, \xi)$$

is a symbol in $S^{m_1+m_2-N}$ for every positive integer N .

THEOREM 2.2. *Let $\sigma \in S^m$. Then the formal adjoint T_{σ}^* is a pseudodifferential operator with symbol σ^* in S^m . Moreover, $\sigma^*(x, \xi)$ has the asymptotic expansion*

$$(2.2) \quad \sigma^*(x, \xi) \sim \sum_{\alpha} \frac{(-i)^{|\alpha|}}{\alpha!} \overline{[\partial_x^{\alpha} \partial_{\xi}^{\alpha} \sigma](x, \xi)}.$$

Here, formula (2.2) means that

$$\sigma^* - \sum_{|\alpha| < N} \frac{(-i)^{|\alpha|}}{\alpha!} \overline{[\partial_x^{\alpha} \partial_{\xi}^{\alpha} \sigma]}$$

is a symbol in S^{m-N} for every positive integer N .

Theorems 2.1 and 2.2 are special cases of Theorem 1.7, pp. 59–60 of Kumano-go [3].

As an application of Theorems 2.1 and 2.2, we give the following theorem.

THEOREM 2.3. *Let T_{σ_1} and T_{σ_2} be elliptic pseudodifferential operators of orders m_1 and m_2 respectively. Then $T_{\sigma_1}T_{\sigma_2}$ and $T_{\sigma_1}^*$ are elliptic pseudodifferential operators of orders $m_1 + m_2$ and m_1 respectively.*

PROOF. Since T_{σ_1} and T_{σ_2} are elliptic, we can find positive constants C_1, C_2 , and R_1 such that

$$\begin{aligned} |\sigma(x, \xi)| &\geq |\sigma_1(x, \xi)\sigma_2(x, \xi)| - |\sigma(x, \xi) - \sigma_1(x, \xi)\sigma_2(x, \xi)| \\ &\geq C_1(1 + |\xi|)^{m_1+m_2} - C_2(1 + |\xi|)^{m_1+m_2-1} \\ &= (1 + |\xi|)^{m_1+m_2}\{C_1 - C_2(1 + |\xi|)^{-1}\}, \quad |\xi| \geq R_1. \end{aligned}$$

Now we can choose $|\xi|$ large enough, say $|\xi| \geq R_2$, so that $C_1 - C_2(1 + |\xi|)^{-1} \geq \frac{1}{2}C_1$. Hence,

$$|\sigma(x, \xi)| \geq \frac{1}{2}C_1(1 + |\xi|)^{m_1+m_2}$$

for $|\xi| \geq R$, where $R = \max(R_1, R_2)$. This proves that $T_{\sigma_1}T_{\sigma_2}$ is elliptic. The proof of the ellipticity of $T_{\sigma_1}^*$ is similar and hence omitted.

The formal adjoint allows us to extend the definition of T_σ from $\mathcal{S}$ to the conjugate linear dual $\mathcal{S}^*$ of $\mathcal{S}$, i.e., the space of tempered distributions. To wit, take any $u \in \mathcal{S}^*$ and we can define $T_\sigma u$ by

$$(T_\sigma u)(\phi) = u(T_\sigma^* \phi)$$

for all ϕ in $\mathcal{S}$. A very important result is the following theorem.

THEOREM 2.4. *Let $\sigma \in S^m$. Then $T_\sigma: H^{s+m,p} \rightarrow H^{s,p}$ is a bounded linear operator for $-\infty < s < \infty$ and $1 < p < \infty$.*

$H^{s,p}$ is the L^p Sobolev space of order s . See Chapter 2, §4 of Schechter [5] for a discussion of these spaces. For a proof of Theorem 2.4, take $\omega(x) \equiv 1$ in Theorem 2.1 of Wong [8].

3. The minimal and maximal operators. Let $\sigma \in S^m$. Then T_σ can be considered as an operator on $L^p(\mathbf{R}^n)$ with domain $\mathcal{S}$. However, it is not closed. Fortunately, it is closable. Hence, it has a closed extension. See Chapter 12, §6 of Schechter [6].

PROPOSITION 3.1. *The operator T_σ defined on $L^p(\mathbf{R}^n)$ with domain $\mathcal{S}$ is closable.*

PROOF. Let $\{\phi_k\}$ be a sequence of functions $\mathcal{S}$ such that $\phi_k \rightarrow 0$, $T_\sigma \phi_k \rightarrow f$ in $L^p(\mathbf{R}^n)$ as $k \rightarrow \infty$. Then for any function ψ in $\mathcal{S}$, we have

$$(T_\sigma \phi_k, \psi) = (\phi_k, T_\sigma^* \psi),$$

where T_σ^* is the formal adjoint constructed in Theorem 2.2. Let $k \rightarrow \infty$. Then we have $(f, \psi) = 0$ for all functions ψ in $\mathcal{S}$. Since $\mathcal{S}$ is dense in $L^p(\mathbf{R}^n)$, it follows that $f = 0$.

REMARK 3.2. A consequence of Proposition 3.1 is that $T_\sigma: \mathcal{S} \rightarrow L^p(\mathbf{R}^n)$ has a closed extension in $L^p(\mathbf{R}^n)$. We denote the smallest such by $T_{\sigma, \min}$ and call it the minimal operator of T_σ . It can be shown easily that the domain $\mathcal{D}(T_{\sigma, \min})$ of $T_{\sigma, \min}$ consists of all functions u in $L^p(\mathbf{R}^n)$ for which a sequence $\{\phi_k\}$ in $\mathcal{S}$ can be found such that $\phi_k \rightarrow u$ in $L^p(\mathbf{R}^n)$ and $T_\sigma \phi_k \rightarrow f$ for some f in $L^p(\mathbf{R}^n)$. Moreover, $T_{\sigma, \min} u = f$. See again [6, Chapter 12, §6].

We say that a function u in $L^p(\mathbf{R}^n)$ lies in $\mathcal{D}(T_{\sigma,\max})$ and $T_{\sigma,\max}u = f$ if and only if

$$(u, T_{\sigma}^* \phi) = (f, \phi)$$

for all ϕ in $\mathcal{S}$.

REMARK 3.3. It can be shown easily that $T_{\sigma,\max}$ is a closed linear operator on $L^p(\mathbf{R}^n)$ with domain $\mathcal{D}(T_{\sigma,\max})$ containing $\mathcal{S}$. We call $T_{\sigma,\max}$ the maximal operator of T_{σ} . It is clear that $\mathcal{D}(T_{\sigma,\max})$ consists of all functions u in $L^p(\mathbf{R}^n)$ for which $T_{\sigma}u$ is in $L^p(\mathbf{R}^n)$. Furthermore, $T_{\sigma,\max}u = T_{\sigma}u$.

The main result in this section is that $T_{\sigma,\min} = T_{\sigma,\max}$ if σ is elliptic. To this end, we recall a very important result in the theory of pseudodifferential operators.

THEOREM 3.4. *Let σ be an elliptic symbol of order m . Then T_{σ} has a parametrix T_{τ} with $\tau \in S^{-m}$. More precisely,*

$$(3.1) \quad T_{\tau}T_{\sigma} = I + R, \quad T_{\sigma}T_{\tau} = I + S$$

where I is the identity, R and S are pseudodifferential operators with symbols in $\bigcap_{k=-\infty}^{\infty} S^k$.

For a proof of Theorem 3.4, see Kumano-go [3, pp. 83–85].

We can now give more precise descriptions of $T_{\sigma,\min}$ and $T_{\sigma,\max}$ when σ is elliptic. First, we consider $T_{\sigma,\min}$.

THEOREM 3.5. *Let σ be an elliptic symbol of order m . Then $\mathcal{D}(T_{\sigma,\min}) = H^{m,p}$ and $T_{\sigma,\min}u = T_{\sigma}u$ for all u in $H^{m,p}$.*

PROOF. The theorem follows easily from the estimate

$$(3.2) \quad \|\phi\|_{m,p} \leq C(\|T_{\sigma}\phi\|_{0,p} + \|\phi\|_{0,p}) \leq C'\|\phi\|_{m,p}, \quad \phi \in \mathcal{S}.$$

Indeed, if $u \in H^{m,p}$, then we can take a sequence $\{\phi_k\}$ of functions in $\mathcal{S}$ such that $\phi_k \rightarrow u$ in $H^{m,p}$. Hence, by (3.2), $\{T_{\sigma}\phi_k\}$ and $\{\phi_k\}$ are Cauchy sequences in $L^p(\mathbf{R}^n)$. So $\phi_k \rightarrow u$ and $T_{\sigma}\phi_k \rightarrow f$ for some u and f in $L^p(\mathbf{R}^n)$. Hence $u \in \mathcal{D}(T_{\sigma,\min})$ and $T_{\sigma,\min}u = f$. On the other hand, if $u \in \mathcal{D}(T_{\sigma,\min})$, then we can find a sequence $\{\phi_k\}$ in $\mathcal{S}$ such that $\phi_k \rightarrow u$ in $L^p(\mathbf{R}^n)$ and $T_{\sigma}\phi_k \rightarrow f$ for some f in $L^p(\mathbf{R}^n)$. Hence, $\{\phi_k\}$ and $\{T_{\sigma}\phi_k\}$ are Cauchy sequences in $L^p(\mathbf{R}^n)$. So, by (3.2), $\{\phi_k\}$ is a Cauchy sequence in $H^{m,p}$. Since $H^{m,p}$ is complete, $\phi_k \rightarrow v$ for some v in $H^{m,p}$. Suppose $m \geq 0$. Then the inclusion map $H^{m,p} \rightarrow L^p(\mathbf{R}^n)$ is continuous. Thus, $\phi_k \rightarrow v$ in $L^p(\mathbf{R}^n)$. Hence $u = v$, and consequently lies in $H^{m,p}$. If $m < 0$, then the inclusion map $L^p(\mathbf{R}^n) \rightarrow H^{m,p}$ is continuous. Thus, $\phi_k \rightarrow u$ in $H^{m,p}$. Hence $u = v$, and consequently lies in $H^{m,p}$ as well. It remains to prove the inequalities (3.2). First, note that the second inequality follows from Theorem 2.4. To prove the first inequality, note that (3.1) implies that

$$\phi = T_{\tau}T_{\sigma}\phi - R\phi$$

for all ϕ in $\mathcal{S}$. Since $\tau \in S^{-m}$ and R has symbol in S^{-m} , it follows from Theorem 2.4 again that

$$\|\phi\|_{m,p} \leq C(\|T_{\sigma}\phi\|_{0,p} + \|\phi\|_{0,p})$$

for all ϕ in $\mathcal{S}$.

We can now prove the following result.

THEOREM 3.6. *Let σ be an elliptic symbol of order m . Then $T_{\sigma,\min} = T_{\sigma,\max}$.*

PROOF. Since $T_{\sigma,\min}$ is the smallest closed extension of $T_\sigma: \mathcal{S} \rightarrow L^p(\mathbf{R}^n)$ in $L^p(\mathbf{R}^n)$, it follows from Theorem 3.5 that it is sufficient to prove that $\mathcal{D}(T_{\sigma,\max}) = H^{m,p}$. Obviously, $H^{m,p} \subseteq \mathcal{D}(T_{\sigma,\max})$ by Remark 3.3. To prove the other inclusion, let $u \in \mathcal{D}(T_{\sigma,\max})$. Then by (3.1), $u = T_\tau T_\sigma u - Ru$. Since $\tau \in S^{-m}$ and $T_\sigma u \in L^p(\mathbf{R}^n)$, we have $T_\tau T_\sigma u \in H^{m,p}$. Since R has symbol in S^{-m} and $u \in L^p(\mathbf{R}^n)$, we also have $Ru \in H^{m,p}$. Hence, $u \in H^{m,p}$.

4. Relative compactness. Let q be a measurable function defined on $\mathbf{R}^n$. Denote by $\mathcal{D}(q)$ the set of all functions u in $L^p(\mathbf{R}^n)$ for which qu is also in $L^p(\mathbf{R}^n)$. Then it can be shown easily that the multiplication operator $q: \mathcal{D}(q) \rightarrow L^p(\mathbf{R}^n)$ given by $u \rightarrow qu$ is a closed operator in $L^p(\mathbf{R}^n)$.

For any elliptic symbol σ of positive order m , we know from Theorem 2.3 that T_σ^k is an elliptic pseudodifferential operator of order km for any positive integer k . We say that q is T_σ^k compact if and only if $H^{km,p} \subseteq \mathcal{D}(q)$ and the multiplication operator $q: H^{km,p} \rightarrow L^p(\mathbf{R}^n)$ is a compact operator. Using Theorems 2.1 and 4.1 in Chapter 6 of Schechter [5], we have the following result.

THEOREM 4.1. *Let T_σ be an elliptic pseudodifferential operator of positive order m and q a measurable function on $\mathbf{R}^n$. Then, in order for q to be T_σ^k compact for some $k \geq 1$, it is sufficient that*

$$(4.1) \quad \sup_y \int_{|x-y|<1} |q(x)|^p dx < \infty,$$

and

$$(4.2) \quad \lim_{|y| \rightarrow \infty} \int_{|x-y|<1} |q(x)|^p dx = 0.$$

We shall prove in §5 that these conditions are also necessary when the symbol σ is independent of x .

5. Schrödinger-type operators. Let $m \in (-\infty, \infty)$. We denote by S_0^m the set of all C^∞ functions σ on $\mathbf{R}^n$ such that for any multi-index α , there exists a constant $C_\alpha > 0$ satisfying

$$|(D^\alpha \sigma)(\xi)| \leq C_\alpha (1 + |\xi|)^{m-|\alpha|}$$

for all $\xi \in \mathbf{R}^n$. Obviously, S_0^m is a subset of S^m . Hence, everything we have stated for S^m is also valid for S_0^m .

In this section, we are mainly interested in operators of the form $T_\sigma^k + q$, where $\sigma \in S_0^m$ is elliptic of positive order m , k is a positive integer, and q is a measurable function on $\mathbf{R}^n$. We call such operators Schrödinger-type operators. See the Introduction (i.e., §1) for the importance of these operators.

THEOREM 5.1. *Let T_σ be an elliptic pseudodifferential operator with symbol in $S_0^m, m > 0$; and let q be a measurable function on $\mathbf{R}^n$. Then in order for q to be compact relative to some positive integral power T_σ^k of T_σ , it is necessary and sufficient that (4.1) and (4.2) are valid.*

PROOF. Of course, in view of Theorem 4.1, we need only prove necessity. Since $\sigma \in S_0^m$, it follows that $T_\sigma^k = T_\tau$, where $\tau(\xi) = \sigma(\xi)^k$. If q is compact relative to T_τ ,

then $H^{km,p} \subseteq \mathcal{D}(q)$ and $q: H^{km,p} \rightarrow L^p(\mathbf{R}^n)$ is compact, hence bounded. Since, by (3.2), $\|\phi\|_{km,p}$ is equivalent to $\|T_\tau \phi\|_{0,p} + \|\phi\|_{0,p}$, we can find a constant $C > 0$ such that

$$(5.1) \quad \|q\phi\|_{0,p} \leq C(\|T_\tau \phi\|_{0,p} + \|\phi\|_{0,p})$$

for all ϕ in $\mathcal{S}$. Let $\psi \in C_0^\infty(\mathbf{R}^n)$ be such that $\psi(x) \geq 0$ for all x in $\mathbf{R}^n$, and $\psi(x) \geq 1$ for $|x| \leq 1$. Define ψ_y by $\psi_y(x) = \psi(x - y)$, for any fixed but arbitrary y in $\mathbf{R}^n$. Then $\psi_y \in C_0^\infty(\mathbf{R}^n)$ and an easy computation shows that

$$(5.2) \quad (T_\tau \psi_y)(x) = (T_\tau \psi)(x - y).$$

Hence, by (5.1)

$$(5.3) \quad \begin{aligned} \|q\psi_y\|_{0,p} &\leq C(\|T_\tau \psi_y\|_{0,p} + \|\psi_y\|_{0,p}) \\ &= C(\|T_\tau \psi\|_{0,p} + \|\psi\|_{0,p}). \end{aligned}$$

But,

$$\|q\psi_y\|_{0,p} \geq \left\{ \int_{|x-y|<1} |q(x)\psi_y(x)|^p dx \right\}^{1/p} \geq \left\{ \int_{|x-y|<1} |q(x)|^p dx \right\}^{1/p}.$$

Hence, by (5.3), (4.1) is proved. To prove (4.2), suppose that it is not true. Then there exists a $\delta > 0$ such that

$$\int_{|x-y_j|<1} |q(x)|^p dx > \delta$$

for some sequence $\{y_j\}$ in $\mathbf{R}^n$. Let ψ be as before and define ψ_j by $\psi_j(x) = \psi(x - y_j)$ for $j = 1, 2, \dots$. Then by (3.2) and (5.2),

$$\begin{aligned} \|\psi_j\|_{km,p} &\sim \|\psi_j\|_{0,p} + \|T_\tau \psi_j\|_{0,p} \\ &= \|\psi\|_{0,p} + \|T_\tau \psi\|_{0,p}. \end{aligned}$$

Since $q: H^{km,p} \rightarrow L^p(\mathbf{R}^n)$ is compact, it follows that there exists a subsequence of $\{\psi_j\}$, again denoted by $\{\psi_j\}$, such that $q\psi_j \rightarrow f$, for some f in $L^p(\mathbf{R}^n)$. But,

$$\int_{\mathbf{R}^n} |q(x)\psi_j(x)|^p dx \geq \int_{|x-y_j|<1} |q(x)|^p dx \geq \delta$$

for $j = 1, 2, \dots$. Hence,

$$(5.4) \quad \|f\|_{0,p} \geq \delta.$$

But on any unit cube Q in $\mathbf{R}^n$, $\psi_j(x) = 0$ if j is large enough. So, $q(x)\psi_j(x) = 0$ for any x in Q . Hence, $\lim_{j \rightarrow \infty} \int_Q |q(x)\psi_j(x)|^p dx = 0$, and consequently, $\int_Q |f(x)|^p dx = 0$. By partitioning $\mathbf{R}^n$ into a grid of unit cubes, we see that $\int_{\mathbf{R}^n} |f(x)|^p dx = 0$. This contradicts (5.4).

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