A SAMPLING THEOREM FOR ANALYTIC FUNCTIONS

J. L. SCHIFF AND W. J. WALKER

ABSTRACT. An analogue to the Shannon sampling theorem is obtained for an analytic function sampled on a circle.

1. Introduction. Let f be analytic on a domain Ω and let D be a disk centered at z_0 with $\overline{D} \subset \Omega$. The sampling theorem presented here gives a surprising formula to recapture f exactly from its values at z_0 and at a countable set of points on ∂D . The expansion obtained for f is valid in the largest open disk center z_0 which is contained in Ω .

Without loss of generality we take $z_0 = 0$ and suppose that f has the Taylor expansion

$$f(z) = \sum_{m=0}^{\infty} a_m z^m = \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} z^m, \quad |z| < R.$$

The theorem gives a precise formula for each a_m (and hence for f) constructed from the countable set of values of f . The Shannon sampling theorem recently surveyed in [1] gives a formula to recapture a certain class of analytic function from its values at a countable set of points on the real axis. In the practical application of a sampling theorem only a finite set of data is sampled and the formula is truncated. The example in §3 demonstrates how the theorem on the disk can give a very precise approximation with a small number of sample points.

2. The sampling theorem. In order to describe the method, we need the Möbius function μ , which is defined as follows: Let j be a positive integer:

- (i) If $j = 1$, then $\mu(j) = 1$.
- (ii) $\mu(j) = 0$ if there is a prime p such that $p^2 | j$.
- (iii) If j has the prime factorization $j = p_1 p_2 \cdots p_\lambda$ where the primes are distinct, then $\mu(j) = (-1)^\lambda$.

The proof of the sampling theorem will require the following lemmas.

LEMMA 1. *The Möbius function μ has the property that*

$$\sum_{d|h} \mu(d) = \begin{cases} 1 & \text{for } h = 1, \\ 0 & \text{for } h > 1. \end{cases}$$

Received by the editors January 2, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30B99; Secondary 30E10.

©1987 American Mathematical Society
 0002-9939/87 \$1.00 + \$.25 per page

PROOF. See [3, p. 114].

The next lemma first appeared in the work of J. L. Walsh [4]. We include the proof for completeness.

LEMMA 2. *Let f be analytic in the disk $|z| < R$. Let k be a positive integer and set $\omega = e^{2\pi i/k}$. Also fix ζ , $|\zeta| < R$, and define*

$$h_1(\zeta) = f(\zeta) - f(0),$$

$$h_k(\zeta) = f(\zeta) + f(\omega\zeta) + f(\omega^2\zeta) + \cdots + f(\omega^{k-1}\zeta) - kf(0), \quad k \geq 2.$$

Then

$$h_k(\zeta) = k \left[\frac{f^{(k)}(0)}{k!} \zeta^k + \frac{f^{(2k)}(0)}{(2k)!} \zeta^{2k} + \cdots + \frac{f^{(nk)}(0)}{(nk)!} \zeta^{nk} + \cdots \right].$$

PROOF. For $k = 1$ the proof is immediate from the Taylor series for f . Now suppose $k \geq 2$ and that n is a multiple of k . Then

$$1 + \omega^n + \omega^{2n} + \cdots + \omega^{(k-1)n} = k.$$

If n is not a multiple of k , then

$$1 + \omega^n + \omega^{2n} + \cdots + \omega^{(k-1)n} = (1 - \omega^{kn}) / (1 - \omega^n) = 0.$$

The Taylor expansion

$$f(z) = f(0) + \frac{f^{(1)}(0)}{1!} z + \cdots + \frac{f^{(n)}(0)}{n!} z^n + \cdots$$

can now be applied to each term of $h_k(\zeta)$. The coefficient of ζ^n is

$$[1 + \omega^n + \omega^{2n} + \cdots + \omega^{(k-1)n}] \frac{f^{(n)}(0)}{n!}$$

and the result follows by evaluating this expression.

We now fix an initial point ζ and define the set of points S at which f is to be sampled as follows:

(i) $0 \in S$.

(ii) For each positive integer k let $\omega = e^{2\pi i/k}$; then the k equally spaced points $\zeta, \omega\zeta, \omega^2\zeta, \dots, \omega^{k-1}\zeta$, belong to S .

If f is given at each point of S , then $h_k(\zeta)$ is given for $k \geq 1$. For convenience we set $H_k(\zeta) = h_k(\zeta)/k$. We can now state the sampling theorem.

THEOREM 3. *Suppose that f is given at each point of the set S . Then for each positive integer m , $a_m = f^{(m)}(0)/m!$ is determined exactly by the formula*

$$(1) \quad a_m = \frac{1}{\zeta^m} \sum_{i=1}^{\infty} \mu(i) H_{im}(\zeta).$$

Hence $f(z) = f(0) + \sum_{m=1}^{\infty} a_m z^m$ is determined exactly.

PROOF. We need only verify the formula for a_m . To do this we use Lemma 2 to expand each term on the right-hand side of (1) so that the right-hand side can be viewed as a double series:

$$\begin{aligned} \frac{\mu(1)}{\xi^m} H_m(\xi) &= \frac{\mu(1)}{\xi^m} \left[\frac{f^{(m)}(0)}{m!} \xi^m + \frac{f^{(2m)}(0)}{(2m)!} \xi^{2m} + \frac{f^{(3m)}(0)}{(3m)!} \xi^{3m} + \frac{f^{(4m)}(0)}{(4m)!} \xi^{4m} + \dots \right], \\ \frac{\mu(2)}{\xi^m} H_{2m}(\xi) &= \frac{\mu(2)}{\xi^m} \left[\frac{f^{(2m)}(0)}{(2m)!} \xi^{2m} + \frac{f^{(4m)}(0)}{(4m)!} \xi^{4m} + \dots \right], \\ \frac{\mu(3)}{\xi^m} H_{3m}(\xi) &= \frac{\mu(3)}{\xi^m} \left[\frac{f^{(3m)}(0)}{(3m)!} \xi^{3m} + \dots \right], \\ &\dots \dots \dots \\ \frac{\mu(j)}{\xi^m} H_{jm}(\xi) &= \frac{\mu(j)}{\xi^m} \left[\dots \dots \dots + \frac{f^{(jm)}(0)}{(jm)!} \xi^{jm} \right]. \end{aligned}$$

We now compute coefficients by adding columns of this array. We denote the j th column by C_j . Thus,

$$\begin{aligned} C_1 &= \frac{\mu(1)}{\xi^m} \frac{f^{(m)}(0)}{m!} \xi^m = \frac{f^{(m)}(0)}{m!} = a_m, \\ C_2 &= \frac{1}{\xi^m} (\mu(1) + \mu(2)) \frac{f^{(2m)}(0)}{(2m)!} \xi^{2m} = 0. \end{aligned}$$

In general for $j \geq 2$ we can apply Lemma 1 to show

$$C_j = \frac{1}{\xi^m} \left(\sum_{d|j} \mu(d) \right) \frac{f^{(jm)}(0)}{(jm)!} \xi^{jm} = 0.$$

Then $\sum_{j=1}^{\infty} C_j = a_m$.

Thus the iterated series calculated by summing columns first converges to a_m . The iterated series calculated by summing rows first will also converge to a_m if we can show that the double series is absolutely convergent. To do this we take the absolute value of each term in the array and sum columns. For $j \geq 2$ the j th column becomes

$$c_j = \frac{1}{|\xi^m|} \left(\sum_{d|j} |\mu(d)| \right) \frac{|f^{(jm)}(0)|}{(jm)!} |\xi|^{jm} \leq \frac{1}{|\xi^m|} j \frac{|f^{(jm)}(0)|}{(jm)!} |\xi|^{jm}.$$

Then

$$\sum_{j=1}^{\infty} c_j \leq \frac{1}{|\xi^m|} \sum_{j=1}^{\infty} j \frac{|f^{(jm)}(0)|}{(jm)!} |\xi|^{jm}.$$

The right-hand series is convergent since $|\xi|$ is less than the radius of convergence of the Taylor series. This completes the proof of the absolute convergence of the double series. Hence the iterated series, summing by rows first,

$$(\xi^m)^{-1} \sum_{i=1}^{\infty} \mu(i) H_{im}(\xi),$$

also converges to a_m , which proves the theorem.

3. Applications. In order to effectively approximate each Taylor coefficient, one must truncate the infinite series $\sum_{i=1}^{\infty} \mu(i) H_{im}(\zeta)$ at some stage. In preliminary computer studies, we have used the preceding algorithm out to $i = 10$, yielding

$$a_m \approx H_{1m}(\zeta) - H_{2m}(\zeta) - H_{3m}(\zeta) - H_{5m}(\zeta) + H_{6m}(\zeta) - H_{7m}(\zeta) + H_{10m}(\zeta),$$

$m = 1, 2, 3, \dots$. The number of points sampled can thus be kept relatively low, and yet highly precise estimates can still be achieved. For example, below is a table of estimates (for $i = 10$) to the Taylor coefficients $a_m = 1/m!$ of $f(z) = e^z$, $m = 1, \dots, 5$.

	Estimate	Actual
$m = 1$	$1.00000012 + 0.00000018 i$	1
$m = 2$	$0.49999991 + 0.00000001 i$	0.5
$m = 3$	$0.16666642 + 0.00000003 i$	0.166...
$m = 4$	$0.04166676 + 0.00000001 i$	0.04166...
$m = 5$	$0.00833333 - 0.00000010 i$	0.00833...

The points sampled were on the unit circle, with the initial point $\zeta = 1$.

If in a practical situation the value at the origin is unknown, then an estimate for it can be obtained from the values on the boundary circle using the mean value property (cf. [2]).

REFERENCES

1. J. R. Higgins, *Five short stories about the cardinal series*, Bull. Amer. Math. Soc. (N.S.) **12** (1985), 45–89.
2. H. Niederreiter, *Quasi-monte carlo methods and pseudo-random numbers*, Bull. Amer. Math. Soc. **84** (1978), 957–1041.
3. B. L. Van der Waerden, *Modern algebra*, vol. 1, Ungar, New York, 1953.
4. J. L. Walsh, *A mean value theorem for polynomials and harmonic polynomials*, Bull. Amer. Math. Soc. **42** (1936), 923–930.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF AUCKLAND, AUCKLAND, NEW ZEALAND