QUOTIENT RINGS OF RINGS GENERATED BY FAITHFUL CYCLIC MODULES

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ABSTRACT. A ring R is said to be generated by faithful cyclics (right finitely pseudo-Frobenius), denoted by right GFC (FPF), if every faithful cyclic (finitely generated) right R -module generates the category of right R -modules. A fundamental result in FPF ring theory, due to S. Page, is that if R is a right nonsingular right FPF ring, then $Q_r(R)$ is FPF. In this paper we generalize this result by providing a necessary and sufficient condition for a right nonsingular right GFC ring to have an FPF maximal right quotient ring.

Introduction. The class of right GFC rings includes right FPF rings, commutative rings (thus every ring has a GFC subring—its center), strongly regular rings, and regular right continuous rings of bounded index [3, Proposition 1.2 and Corollary 3.7]. To see that the class of right GFC rings is a nontrivial generalization of the class of right FPF rings, one need only recall that a commutative domain is FPF if and only if it is Prüfer, and a strongly regular ring is FPF if and only if it is selfinjective [8].

All rings are associative with unity. The following notation is used: R is a ring, $Q_r(R)$ is the maximal right quotient ring of R , $\text{Tr}_R(M)$ is the trace ideal of M in R , $l_R(X)$ [$r_R(X)$] is the left [right] annihilator of X in R . Two-sided concepts which are unmodified by the words “left” or “right” mean both sides are satisfied (e.g., R is GFC means R is right and left GFC). From [5 and 12] a ring R is (*quasi*-)Baer if the right annihilator of every (ideal) nonempty subset of R is a direct summand of R . We note that any right nonsingular right FPF ring is quasi-Baer [7, p. 168; 8, Theorem 3.3]. The basic references are [1, 8, 10, and 17]. The following lemma appears in [3, Lemma 1.1] but is included here since it will be used repeatedly throughout the sequel.

LEMMA 1. *R is right GFC if and only if for every right ideal X of R such that X contains no nonzero ideal then $l(X)R = R$.*

The following theorem generalizes [8, Corollary 5.11D and 3, Theorem 3.5].

THEOREM 2. *Let R be a right selfinjective ring. Then R is right GFC if and only if R is right FPF.*

PROOF. Assume R is right GFC. Let $M = \sum_{i=1}^n x_i R$ be a finitely generated faithful R -module, where $\{x_i | i = 1, 2, \dots, n\}$ is a set of generators of M . Since M is faithful, $\bigcap_{i=1}^n r(x_i)$ contains no nonzero ideals. By Lemma 1, [1, p. 113, Exercise 8

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and **17**, p. 275, Proposition 2.2],

$$R = \underline{l} \left(\bigcap_{i=1}^n \underline{r}(x_i) \right) R = \sum_{i=1}^n \underline{l}(\underline{r}(x_i))R = \sum_{i=1}^n \text{Tr}_R(x_i R).$$

But R is right selfinjective, so $\text{Tr}_R(x_i R) \subseteq \text{Tr}_R(M)$ for all $i = 1, 2, \dots, n$. Thus $\text{Tr}_R(M) = R$. From [1, Proposition 8.21], R is right FPF. The converse is obvious.

THEOREM 3. *Let R be a right GFC ring such that whenever I is a nonzero ideal of R , then $IQ_r(R)$ (i.e., the right ideal of $Q_r(R)$ generated by I) contains a nonzero ideal of $Q_r(R)$. If $Q_r(R)$ is right selfinjective, then $Q_r(R)$ is right FPF.*

PROOF. Let M be a faithful cyclic $Q_r(R)$ -module, $g: Q_r(R) \rightarrow M$ a $Q_r(R)$ -epimorphism, and $h: R \rightarrow Q_r(R)$ the inclusion homomorphism. Then $gh(R) = N$ is an R -module. Let $A = \underline{r}_R(N)$. If $A \neq 0$, there exists a nonzero ideal I of $Q_r(R)$ such that $I \subseteq AQ_r(R)$. Let $H = \{m \in M \mid mI = 0\}$. Now H is a $Q_r(R)$ -submodule of M . Hence $H = M$ since $g(1) \in N \subseteq H$. Contradiction! Thus N is a generator in mod- R . Hence there exists an R -epimorphism $f: \bigoplus_{i=1}^k N_i \rightarrow R$ where N_i is R -isomorphic to N . From [14, pp. 94–95], there exists a $Q_r(R)$ -homomorphism $\bar{f}: \bigoplus_{i=1}^k M_i \rightarrow Q_r(R)$ which extends f , where M_i is $Q_r(R)$ -isomorphic to M . Therefore, there exists $x \in \bigoplus_{i=1}^k N_i$ such that $f(x) = \bar{f}(x) = 1$. Hence $\bar{f}(\bigoplus_{i=1}^k M_i) = Q_r(R)$. Consequently, $Q_r(R)$ is right GFC. By Theorem 2, $Q_r(R)$ is right FPF.

COROLLARY 4. *Let R be a right nonsingular right GFC ring. Then the following statements are equivalent.*

- (i) $Q_r(R)$ is right GFC.
- (ii) $Q_r(R)$ is FPF.
- (iii) If I is a nonzero ideal of R , then $IQ_r(R)$ contains a nonzero ideal of $Q_r(R)$.
- (iv) If I is a nonzero ideal of R , then $IQ_r(R)$ is an essential extension of an ideal of $Q_r(R)$.

PROOF. (i) $\Rightarrow$ (ii) follows from Theorem 2 and [8, Corollary 3.14B].

(ii) $\Rightarrow$ (iii). Suppose $IQ_r(R)$ contains no nonzero ideals of $Q_r(R)$. Let $0 \neq x \in \underline{l}_Q(IQ_r(R)) \cap (IQ_r(R))$ [3, Proposition 1.4]. Since I is essential in $IQ_r(R)$ as R -modules, there exists $s \in R$ such that $0 \neq xs \in I$. Then $xs \in \underline{l}_R(I) \cap I$. But R is semiprime [3, Proposition 1.4]. Contradiction!

(iii) $\Rightarrow$ (i). This is a consequence of Theorem 3.

(iii) $\Leftrightarrow$ (iv). Suppose condition (iii) holds. Let X be the maximal ideal of $Q_r(R)$ contained in $IQ_r(R)$. Assume X is not essential in $IQ_r(R)$. Let Y be a complement of X in $IQ_r(R)$. By [3, Proposition 2.6] there exists a central idempotent $b \in Q_r(R)$ such that $Y \subseteq bQ_r(R)$ and $X \cap bQ_r(R) = 0$. Let $B = bQ_r(R) \cap I$. Then B is a nonzero ideal in R . Hence $BQ_r(R)$ contains a nonzero ideal of $Q_r(R)$. But this contradicts the maximality of X . Therefore, X is essential in $IQ_r(R)$. The converse is obvious.

We note that if R is a right nonsingular ring (not necessarily right GFC) which satisfies condition (iii) of Corollary 4 for which $Q_r(R)$ is FPF, then R satisfies condition (iv). For example, if R is a right Ore domain which is not right GFC (e.g., a simple Ore domain which is not a division ring [6; 3, Corollary 1.3]), then R is a nonsingular ring which satisfies condition (iii) of Corollary 4 for which $Q_r(R)$ is

FPF. The ring R of 2×2 upper triangular matrices over a field provides an example of a right nonsingular ring which does not satisfy condition (iii) of Corollary 4 even though $Q_r(R)$ is FPF. Finally from [3, Proposition 1.4 and Corollary 3.6], we see that a right nonsingular right GFC quasi-Baer (e.g., a right nonsingular right FPF) ring satisfies condition (iii) of Corollary 4. However, there are right nonsingular right GFC rings which satisfy condition (iii) of Corollary 4 but are not quasi-Baer. For example, if R is strongly regular, then R is nonsingular GFC and R satisfies condition (iii) of Corollary 4 since $Q_r(R)$ is strongly regular [17, p. 254]. But there are strongly regular rings which are not quasi-Baer (e.g., any noncomplete Boolean ring [17, pp. 79 and 253]). Hence Corollary 4 generalizes [3, Theorem 2.8 and Corollary 3.6; 8, Theorem 3.5].

COROLLARY 5. *If R is a right GFC ring which satisfies any of the following conditions, then $Q_r(R)$ is FPF.*

- (i) *R is right nonsingular and every nonzero ideal of R contains a nonzero central element of R .*
- (ii) *Every nonzero ideal of R contains a nonzero central idempotent (e.g., biregular rings).*
- (iii) *R is a P.I. ring which is either semiprime or right nonsingular.*

PROOF. Corollary 4 and [10, p. 48, Exercise 14] show that the result is true for condition (i). Clearly condition (ii) is a special case of condition (i). From [3, Proposition 1.4; 9, Corollary 2; and 16, Theorem 2] it follows that condition (iii) is a special case of condition (i).

We note that in [13] S. Kobayashi has characterized the regular right GFC rings satisfying condition (ii) of Corollary 5.

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