THE ASYMPTOTIC BEHAVIOR OF UNIVALENT FUNCTIONS

HU KE AND DONG XINHAN

ABSTRACT. We generalize Bazilevic's Theorem and provide an alternative proof of an important theorem of Hamilton.

1. Introduction. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$. B. G. Eke [2] defined a class of functions $S(\theta_1, \theta_2, \dots, \theta_k)$ as follows: $f \in S$ and there exist $C > 0$, $\delta > 0$, and a sequence $r_n \rightarrow 1$ such that [7]

$$(1.1) \quad (1 - r_n)^{2/k} |f(r_n e^{i\theta_n^{(\nu)}})| \geq C, \quad \nu = 1, 2, \dots, k, \quad n = 1, 2, \dots,$$

for

$$(1.2) \quad \delta < \theta_n^{(\nu)} - \theta_n^{(\mu)} < 2\pi - \delta, \quad 1 \leq \nu < \mu \leq k.$$

The following is from works of Hayman [4] and Eke [2].

THEOREM A. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$, then

$$\lim_{n \rightarrow \infty} \|a_n\| - \|a_{n+1}\| \leq 1$$

except $f \in S(\theta_1, \theta_2)$.

In 1982 Hamilton [3] obtained a beautiful result:

THEOREM B. If $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S(\theta_1, \theta_2)$, then

$$\overline{\lim}_{n \rightarrow \infty} \|a_n\| - \|a_{n+1}\| \leq 1.$$

Here we generalize Bazilevic's Theorem and provide an alternate proof of this important theorem of Hamilton.

2. A generalization of Bazilevic's Theorem. We now define

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \gamma_n z^n,$$

the Grunsky expansion

$$G(z, \zeta) = \log \frac{1/f(z) - 1/f(\zeta)}{1/z - 1/\zeta} = - \sum_{n=1}^{\infty} A_n \left(\frac{1}{\zeta} \right) z^n,$$

and the Faber polynomials for $F(\xi) = 1/f(z)$ with $\xi = 1/z$

$$(2.1) \quad \Phi_n \left(\frac{1}{f(z)} \right) = \frac{z^{-n}}{n} + A_n \left(\frac{1}{z} \right) \quad \text{and} \quad 2\gamma_n = \Phi_n(0).$$

Received by the editors March 10, 1985 and, in revised form, February 24, 1986.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 30C50; Secondary 30C55.

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0002-9939/87 \$1.00 + \$.25 per page

The Grunsky inequality states that

$$(2.2) \quad \sum_{n=1}^{\infty} n \left| \sum_{\nu=1}^k A_n \left(\frac{1}{z_{\nu}} \right) \right|^2 \leq \sum_{\mu, \nu=1}^k \log \frac{1}{|1 - z_{\mu} \bar{z}_{\nu}|}.$$

We have

THEOREM 1. *If $f(z) \in S(\theta_1, \theta_2, \dots, \theta_k)$, then*

$$(2.3) \quad \sum_{n=1}^{\infty} n \left| \gamma_n - \frac{1}{k} \sum_{\nu=1}^k e^{-in\theta_{\nu}} \right|^2 \leq \frac{1}{2} \log \frac{1}{C} + \frac{1}{2k} \sum_{\mu \neq \nu} \log \frac{2}{|e^{i\theta_{\mu}} - e^{i\theta_{\nu}}|^2},$$

where C is determined by (1.1) and θ_{ν} is one of the limit points of $\{\theta_n^{(\nu)}\}$ for each ν ($\nu = 1, 2, \dots, k$).

The case $k = 1$ is due to Bazilevic [1].

PROOF. Notice the following identity:

$$(2.4) \quad \begin{aligned} & \left| \sum_{\nu=1}^k \left(A_n \left(\frac{1}{z_{\nu}} \right) - \frac{\bar{w}_{\nu}^n}{n} \right) \right|^2 \\ &= \left| \sum_{\nu=1}^k A_n \left(\frac{1}{z_{\nu}} \right) \right|^2 - \frac{2}{n} \operatorname{Re} \sum_{\mu, \nu=1}^k A_n \left(\frac{1}{z_{\nu}} \right) w_{\mu}^n + \left| \sum_{\nu=1}^k \frac{w_{\nu}^n}{n} \right|^2, \end{aligned}$$

and use (2.2) and the classical inequality

$$\left| \frac{zf'(z)}{f(z)} \right| \leq \frac{1+|z|}{1-|z|} \quad \text{for } |z| = r_n$$

to deduce

$$(2.5) \quad \begin{aligned} \sum_{n=1}^{\infty} n \left| \sum_{\nu=1}^k A_n \left(\frac{1}{z_{\nu}} \right) - \frac{\bar{z}_{\nu}}{n} \right|^2 &\leq 2 \sum_{\mu, \nu=1}^k \left\{ \log \frac{1}{|1 - z_{\mu} \bar{z}_{\nu}|} + \operatorname{Re} G(z_{\mu}, z_{\nu}) \right\} \\ &\leq 2 \sum_{\nu=1}^k \log \frac{|f'(z_{\nu})||z_{\nu}|^2}{|f(z_{\nu})|^2(1-|z_{\nu}|^2)} \\ &\quad + 2 \sum_{\mu \neq \nu} \log \frac{|1/f(z_{\mu}) - 1/f(z_{\nu})||z_{\mu} z_{\nu}|}{|z_{\mu} - z_{\nu}||1 - z_{\mu} \bar{z}_{\nu}|} \\ &\leq 2 \sum_{\nu=1}^k \log \frac{1}{(1-r_m)^2 |f(z_{\nu})|} \\ &\quad + 2 \sum_{\mu \neq \nu} \log \frac{1/|f(z_{\mu})| + 1/|f(z_{\nu})|}{|z_{\mu} - z_{\nu}||1 - z_{\mu} \bar{z}_{\nu}|} \\ &\leq 2 \sum_{\nu=1}^k \log \frac{(1-r_m)^{2/k}}{C(1-r_m)^2} \\ &\quad + 2 \sum_{\mu \neq \nu} \log \frac{2(1-r_m)^{2/k}}{C|z_{\mu} - z_{\nu}|^2}. \end{aligned}$$

Now as $z_m^{(\nu)} \rightarrow e^{i\theta_\nu}$, A_n tends to $\Phi_n(0) - e^{-in\theta_\nu}/n = 2\gamma_n - e^{-in\theta_\nu}/n$. Thus

$$(2.6) \quad \sum_{n=1}^{\infty} n \left| 2k\gamma_n - 2 \sum_{\nu=1}^k \frac{e^{-in\theta_\nu}}{n} \right|^2 \leq 2k^2 \log \frac{1}{C} + 2 \sum_{\mu \neq \nu} \log \frac{2}{|e^{i\theta_\mu} - e^{i\theta_\nu}|^2}.$$

Dividing by $4k^2$ on both sides we complete the proof of Theorem 1.

THEOREM 2. *For all $0 < p < \infty$, if $f(z) \in S(\theta_1, \theta_2, \dots, \theta_k)$, then*

$$\frac{f(z)}{z} \prod_{\nu=1}^n (1 - ze^{-i\theta_\nu})^{2/k} \in H^p$$

and there exists $F_p(t) \in L[0, 2\pi]$ such that

$$(2.7) \quad f(z) = z \prod_{\nu=1}^k (1 - ze^{-i\theta_\nu})^{-2/k} \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} F_p(t) dt \right\}^{1/p},$$

$z = re^{i\theta}.$

The case $k = 1$ is proved in [5].

PROOF. For $p \in (0, \infty)$, let

$$\begin{aligned} \varphi(z) &= \left\{ \frac{f(z)}{z} \prod_{\nu=1}^k (1 - ze^{-i\theta_\nu})^{2/k} \right\}^{p/2} = \sum_{n=0}^{\infty} D_n(p) z^n, \quad D_0 = 1, \\ \log \varphi(z) &= \frac{p}{2} \sum_{n=1}^{\infty} \left(2\gamma_n - \frac{2}{k} \sum_{\nu=1}^k \frac{e^{-in\theta_\nu}}{n} \right) z^n. \end{aligned}$$

From the Milin [5] inequality and Theorem 1 we conclude

$$(2.8) \quad \begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} |\varphi(z)|^2 d\theta &= \sum_{n=0}^{\infty} |D_n(p)|^2 r^{2n} \\ &\leq \exp \left\{ \frac{p^2}{4} \sum_{n=1}^{\infty} n \left| 2\gamma_n - \frac{2}{k} \sum_{\nu=1}^k \frac{e^{-in\theta_\nu}}{n} \right|^2 r^{2n} \right\} \\ &\leq \exp \left\{ \frac{p^2}{4} \left[2 \log \frac{1}{C} + \frac{2}{k^2} \sum_{\mu \neq \nu} \log \frac{2}{|e^{i\theta_\mu} - e^{i\theta_\nu}|^2} \right] \right\} < \infty. \end{aligned}$$

Consequently,

$$\varphi^2 = \left\{ \frac{f(z)}{z} \prod_{\nu=1}^k (1 - ze^{-i\theta_\nu})^{2/k} \right\}^p \in H^1,$$

and $f(z)$ has the integral representation as in (2.7).

3. An application of Theorem 1. We now prove Theorem B. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S(\theta_1, \theta_2)$. Then

$$\begin{aligned} (1 - ze^{-i\theta_1}) \frac{f(z)}{z} &= 1 + \sum_{n=1}^{\infty} (a_{n+1} - e^{-i\theta_1} a_n) z^n, \\ \log \left[(1 - ze^{-i\theta_1}) \frac{f(z)}{z} \right] &= \sum_{n=1}^{\infty} \left(2\gamma_n - \frac{e^{-in\theta_1}}{n} \right) z^n, \\ \log \left[(1 - ze^{-i\theta_1})(1 - ze^{-i\theta_2}) \frac{f(z)}{z} \right] &= \sum_{n=1}^{\infty} \left(2\gamma_n - \frac{e^{-in\theta_1}}{n} - \frac{e^{-in\theta_2}}{n} \right) z^n. \end{aligned}$$

Replacing z by $\sqrt{r}z$ and using the Milin inequality [6] we obtain

$$\begin{aligned} T_n^2 &= \|a_n\| - \|a_{n+1}\|^2 \leq |a_{n+1} - a_n e^{-i\theta_1}|^2 \\ &\leq r^{-n} \exp \left\{ \sum_{k=1}^n k \left| 2\gamma_k - \frac{e^{-ik\theta_1}}{k} \right|^2 r^k - \sum_{k=1}^n \frac{1}{k} \right\} \\ &\leq r^{-n} \exp \left\{ \sum_{k=1}^{\infty} k \left| 2\gamma_k - \frac{e^{-ik\theta_1}}{k} - \frac{e^{-ik\theta_2}}{k} \right|^2 r^k + \sum_{k=1}^n \frac{1}{k} r^k - \sum_{k=1}^n \frac{1}{k} \right. \\ &\quad \left. + 2 \operatorname{Re} \sum_{k=1}^n r^k e^{ik\theta_2} \left[2\gamma_k - \frac{e^{-ik\theta_1}}{k} - \frac{e^{-ik\theta_2}}{k} \right] \right\} \\ &\leq r^{-n} \exp \left\{ \sum_{k=1}^{\infty} k \left| 2\gamma_k - \frac{e^{-ik\theta_1}}{k} - \frac{e^{-ik\theta_2}}{k} \right|^2 \right. \\ &\quad \left. + 2 \operatorname{Re} \sum_{k=1}^{\infty} r^k e^{ik\theta_2} \left[2\gamma_k - \frac{e^{-ik\theta_1}}{k} - \frac{e^{-ik\theta_2}}{k} \right] \right. \\ &\quad \left. - 2 \operatorname{Re} \sum_{k=n+1}^{\infty} r^k e^{ik\theta_2} \left| 2\gamma_k - \frac{e^{-ik\theta_1}}{k} - \frac{e^{-ik\theta_2}}{k} \right| \right\} \\ &\leq r^{-n} \exp \left\{ \sum_{k=1}^{\infty} k \left| 2\gamma_k - \frac{e^{-ik\theta_1}}{k} - \frac{e^{-ik\theta_2}}{k} \right|^2 \right. \\ &\quad \left. + 2 \log \frac{(1-r)|1 - re^{i(\theta_1-\theta_2)}| |f(re^{i\theta_2})|}{r} \right. \\ &\quad \left. + 2 \sum_{k=n+1}^{\infty} r^k \left| 2\gamma_k - \frac{e^{-ik\theta_1}}{k} - \frac{e^{-ik\theta_2}}{k} \right| \right\}. \end{aligned}$$

By Theorem 1,

$$\sum_{k=1}^{\infty} k \left| 2\gamma_k - \frac{e^{-ik\theta_1}}{k} - \frac{e^{-ik\theta_2}}{k} \right|^2 < \infty.$$

Therefore,

$$\begin{aligned} & \sum_{k=n+1}^{\infty} r^k \left| 2\gamma_k - \frac{e^{-ik\theta_1}}{k} - \frac{e^{-ik\theta_2}}{k} \right| \\ & \leq \left\{ \sum_{k=n+1}^{\infty} k \left| 2\gamma_k - \frac{e^{-ik\theta_1}}{k} - \frac{e^{-ik\theta_2}}{k} \right|^2 \sum_{k=n+1}^{\infty} \frac{r^{2k}}{k} \right\}^{1/2} \leq \left\{ \frac{\varepsilon_n}{n(1-r)} \right\}^{1/2}, \end{aligned}$$

where

$$\varepsilon_n = \sum_{k=n+1}^{\infty} k \left| 2\gamma_k - \frac{e^{-ik\theta_1}}{k} - \frac{e^{-ik\theta_2}}{k} \right|^2.$$

Thus we have

$$\begin{aligned} (3.1) \quad T_n^2 & \leq r^{-n} \exp \left\{ \sum_{k=1}^{\infty} k \left| 2\gamma_k - \frac{e^{-ik\theta_1}}{k} - \frac{e^{-ik\theta_2}}{k} \right|^2 \right. \\ & \quad \left. + 2 \log \frac{(1-r)|1 - re^{i(\theta_2-\theta_1)}| \|f(re^{i\theta_2})\|}{r} + 2 \left[\frac{\varepsilon_n}{n(1-r)} \right]^{1/2} \right\} \\ & \leq r^{-n} \exp \left\{ S_1(r) + \omega_1(r) + 2 \left[\frac{\varepsilon_n}{n(1-r)} \right]^{1/2} \right\}, \end{aligned}$$

where

$$\begin{aligned} S_1(r) & = \sum_{k=1}^{\infty} k \left| 2\gamma_k - \frac{e^{-ik\theta_1}}{k} - \frac{e^{-ik\theta_2}}{k} \right|^2 - \sum_{k=1}^{\infty} k \left| A_k \left(\frac{1}{r} e^{-i\theta_1} \right) - \frac{(re^{-i\theta_2})^k}{k} \right|^2, \\ \omega_1(r) & = \sum_{k=1}^{\infty} k \left| A_k \left(\frac{1}{r} e^{-i\theta_1} \right) - \frac{(re^{-i\theta_2})^k}{k} \right|^2 + 2 \log \frac{(1-r)|1 - re^{i(\theta_2-\theta_1)}| \|f(re^{i\theta_2})\|}{r}. \end{aligned}$$

Since $|f(re^{i\theta_\nu})| \rightarrow \infty$ as $r \rightarrow 1$ ($\nu = 1, 2$), we conclude from (2.5) and Theorem 1 that $S_1(r) \rightarrow 0$ as $r \rightarrow 1$. Applying (2.3) with $k = 1$ we have

$$\sum_{n=1}^{\infty} n \left| A_n \left(\frac{1}{r} e^{-i\theta_1} \right) - \frac{(re^{-i\theta_2})^n}{n} \right|^2 \leq 2 \log \frac{(r/(1-r)) [1/|f(re^{i\theta_1})| + 1/|f(re^{i\theta_2})|]}{(1+r)|e^{i\theta_1} - e^{i\theta_2}|}.$$

Hence,

$$\begin{aligned} \omega_1(r) & \leq 2 \log \frac{|1 - re^{i(\theta_2-\theta_1)}| (1 + |f(re^{i\theta_2})/f(re^{i\theta_1})|)}{(1+r)|e^{i\theta_1} - e^{i\theta_2}|} = 2\delta_1(r), \\ (3.2) \quad T_n^2 & \leq r^{-n} \exp \left\{ S_1(r) + 2\delta_1(r) + 2 \left[\frac{\varepsilon_n}{n(1-r)} \right]^{1/2} \right\}. \end{aligned}$$

Similarly, we have

$$(3.2)' \quad T_n^2 \leq r^{-n} \exp \left\{ S_2(r) + 2\delta_2(r) + 2 \left[\frac{\varepsilon_n}{n(1-r)} \right]^{1/2} \right\},$$

where

$$S_2(r) = \sum_{k=1}^{\infty} k \left| 2\gamma_k - \frac{e^{-i\theta_1}}{k} - \frac{e^{-i\theta_2}}{k} \right|^2 - \sum_{k=1}^{\infty} k \left| A_k \left(\frac{1}{r} e^{-i\theta_2} \right) - \frac{(re^{-i\theta_1})^k}{k} \right|^2,$$

$$\delta_2(r) = \log \frac{|1 - re^{i(\theta_1 - \theta_2)}| (1 + |f(re^{i\theta_1})/f(re^{i\theta_2})|)}{(1+r)|e^{i\theta_1} - e^{i\theta_2}|}.$$

Thus

$$(3.3) \quad T_n^2 \leq r_n^{-n} \exp \left\{ \eta(r) + 2 \left[\frac{\varepsilon_n}{n(1-r_n)} \right]^{1/2} \right\},$$

where $\eta(r) = \min(S_1(r_n) + 2\delta_1(r_n), S_2(r_n) + 2\delta_2(r_n))$, i.e., if $|f(re^{i\theta_1})| \geq |f(re^{i\theta_2})|$, then $\eta(r) = S_1(r) + 2\delta_1(r)$, otherwise $\eta(r) = S_2(r) + 2\delta_2(r)$. Therefore $\eta(r) \rightarrow 0$ as $r \rightarrow 1$. Take $r = 1 - \sqrt{\varepsilon_n}/n$ to deduce

$$(3.4) \quad T_n^2 \leq \left(1 - \frac{\sqrt{\varepsilon_n}}{n} \right)^{-n} \exp\{\eta(r_n) + 2\varepsilon_n^{1/4}\},$$

By Theorem 1, $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. It follows that $\lim_{n \rightarrow \infty} T \leq 1$. The proof is complete.

4. Remark. In view of Theorem 2 it is natural to ask if

$$\frac{f(z)}{z} \prod_{k=1}^n (1 - ze^{-i\theta_k})^{2/k} \in \text{BMO } A$$

as $H_p \supset \text{BMO } A \supset H_\infty$. However, we can find $f \in S(0, \pi)$, for example, not in $\text{BMO } A$.

$C - f(D)$ consists of 4 symmetric curves asymptotic to the OY axis at ∞ . Considering 1/W plan:

Choose a curve such that f is conformal at ± 1 , so will have angular limit. Then there exists

$$\lim_{r \rightarrow 1} (1-r)|f(\pm r)| = \alpha > 0.$$

The cusp can be chosen so that $w_n \rightarrow \infty$ in the cusp $\text{dist}(w_n, C - f(D)) = \infty$. The same will be true for $(1-z^2)f(z) = f_0$. Thus f_0 is not Bloch at $\pm i$, and therefore cannot be $\text{BMO } A$ (since f_0 is univalent near $\pm i$).

This example was provided by the referee.

5. Another remark. The contents of this paper have been presented to the meeting on Function Theory held in Nanjing in May 1985.

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DEPARTMENT OF MATHEMATICS, JIANGXI NORMAL UNIVERSITY, NANCHANG, CHINA

DEPARTMENT OF MATHEMATICS, HUNAN NORMAL UNIVERSITY, CHANGSHA, CHINA