ON THE RELATION BETWEEN THE SEMIGROUP AND ITS INFINITESIMAL GENERATOR

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ABSTRACT. We prove that for a semigroup generator A the solutions of $u'(t) = Au(t)$ and $u(0) = x$ are completely determined by the resolvent of A along an equidistant sequence $(s_0 + nr)_{n \in \mathbb{N}}$, $r \in \mathbb{R}$, $s_0 \in \mathbb{C}$, of points in the resolvent set of A .

Most of the fundamental concepts in the theory of strongly continuous semigroups of bounded linear operators on a Banach space E are based on the relation between the semigroup and the resolvent of its infinitesimal generator. For a given semigroup $(T(t))$ the resolvent $R(\lambda, A)$ of the infinitesimal generator A ,

$$Ax = \lim_{t \rightarrow 0} \frac{1}{t} (T(t)x - x) \quad \text{for } x \in D(A),$$

is given as the Laplace transform of the semigroup $(T(t))$, i.e.

$$R(\lambda, A)x = \int_0^\infty e^{-\lambda t} T(t)x \, dt \quad \text{for every } x \in E$$

and every $\lambda \in \mathbb{C}$ with

$$\begin{aligned} \operatorname{Re} \lambda > w_1 &:= \inf \{w: \text{for every } x \in D(A) \text{ there exist a constant } M(x, w) \\ &\quad \text{such that } \|T(t)x\| \leq M(x, w)e^{wt} \text{ for every } t \geq 0\} \\ &= \inf \left\{ \operatorname{Re} \lambda: \int_0^\infty e^{-\lambda t} T(t)x \, dt \text{ exists for all } x \in E \right\} \end{aligned}$$

(see [4]).

From the point of view of applications to partial differential equations it is more relevant to obtain the solutions $T(\cdot)x$ of the initial value problem

$$(*) \quad u'(t) = Au(t), \quad u(0) = x \in D(A)$$

from the resolvent of the infinitesimal generator A . Well-known formulas are the Yosida approximation

$$T(t)x = \lim_{\lambda \rightarrow \infty} e^{t\lambda AR(\lambda, A)}x \quad (\text{for } x \in E)$$

and the exponential (or Post-Widder inversion-) formula

$$\begin{aligned} T(t)x &= \lim_{n \rightarrow \infty} \left[\frac{n}{t} R\left(\frac{n}{t}, A\right) \right]^n x \\ &= \lim_{n \rightarrow \infty} (-1)^n \frac{1}{k!} \left(\frac{n}{t}\right)^{n+1} R^{(n)}\left(\frac{n}{t}, A\right) x \quad (x \in E). \end{aligned}$$

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In both formulas, for $\lambda \geq \lambda_0$, the powers of $R(\lambda, A)$ have to be known. However, in applications it is for the most part impossible to estimate the powers of the resolvent. For practical purposes the most useful representation of $T(\cdot)x$, $x \in D(A)$, is given by

$$T(t)x = \frac{1}{2\pi i} \lim_{r \rightarrow \infty} \int_{\gamma - ir}^{\gamma + ir} e^{\lambda t} R(\lambda, A)x d\lambda \quad (t > 0),$$

the inversion formula of the Laplace transform. Investigating semigroups with different continuity properties, E. Hille and R. Phillips assumed—in analogy to the inversion formula for the general real Laplace-Stieltjes transform—the constant γ to be greater than $\max\{0, w_0\}$; w_0 being the growth bound of the semigroup (see [2, p. 349]). We remark that for strongly continuous semigroups the condition $\gamma > \max\{0, w_0\}$ can be replaced by $\gamma > w_1$ (the proof given by Widder [6, p. 64ff] carries over to the infinite-dimensional case, see [3]). For eventually normcontinuous semigroups on Banach spaces (as there are compact, differentiable, holomorphic, nilpotent or normcontinuous semigroups) [1], and for positive semigroups on Banach spaces ordered by a normal and generating cone [4], the growth bound w_1 is equal to the spectral bound of A i.e. the equality

$$s(A) := \sup\{\operatorname{Re} \lambda, \lambda \in \sigma(A)\} = w_1$$

holds. Therefore, in these cases $T(t)x$, $x \in D(A)$, can be completely represented in terms of the infinitesimal generator A ($R(\lambda, A)x$ has to be known for $\lambda \in \gamma + i\mathbf{R}$, $\gamma > s(A)$).

In this note we would like to call attention to a representation formula due to E. Phragmén [5, p. 360ff] or [7, p. 166]. Using this formula we will see that the solutions of (*) are completely determined by the resolvent along an equidistant sequence $(s_0 + nr)_{n \in \mathbf{N}}$, $r \in \mathbf{R}$, $s_0 \in \mathbf{C}$, of points in the resolvent set of A .

PHRAGMÉN'S REPRESENTATION THEOREM. *Let A be the generator of a strongly continuous semigroup $(T(t))$ on a Banach space E . Then, for every $x \in D(A)$,*

$$T(t)x = \lim_{\lambda \rightarrow \infty} \lambda \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(n-1)!} e^{n\lambda t} R(n\lambda, A)x.$$

PROOF. For $\lambda \geq w > w_0$ the resolvent $R(\lambda, A)$ is bounded. Therefore, for every $x \in E$,

$$y := \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!} e^{n\lambda t} R(n\lambda, A)x$$

exists. By

$$R(n\lambda, A)x = \int_0^{\infty} e^{-n\lambda s} T(s)x ds$$

we obtain

$$y = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!} e^{n\lambda t} \int_0^{\infty} e^{-n\lambda s} T(s)x ds.$$

Define

$$f_N(s) := \left[\sum_{n=1}^N (-1)^{n+1} \frac{1}{n!} e^{n\lambda(t-s)} \right] T(s)x.$$

Then, as $N \rightarrow \infty$, $f_N(\cdot)$ converges uniformly on compacts to

$$(1 - e^{-e^{\lambda(t-s)}})T(s)x.$$

Hence

$$\begin{aligned} & \int_0^T (1 - e^{-e^{\lambda(t-s)}}) T(s)x \, ds \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!} e^{n\lambda t} \int_0^T e^{-n\lambda s} T(s)x \, ds =: z_T. \end{aligned}$$

By

$$\|z_T - y\| \leq \left(\sum_{n=1}^{\infty} \frac{1}{n!} e^{n\lambda t} \right) \int_T^{\infty} \|e^{-\lambda s} T(s)x\| \, ds \rightarrow 0$$

as $T \rightarrow \infty$ we obtain $y = \int_0^{\infty} (1 - e^{-e^{\lambda(t-s)}}) T(s)x \, ds$. Now, for $0 < \delta < t$,

$$\begin{aligned} \left\| y - \int_0^t T(s)x \, ds \right\| &\leq \left\| \int_0^{t-\delta} e^{-e^{\lambda(t-s)}} T(s)x \, ds \right\| + \int_{t-\delta}^{t+\delta} \|T(s)x\| \, ds \\ &\quad + \left\| \int_{t+\delta}^{\infty} (1 - e^{-e^{\lambda(t-s)}}) T(s)x \, ds \right\|. \end{aligned}$$

Given $\varepsilon > 0$ chose δ such that the second term is less than $\varepsilon/3$. For all $0 \leq s \leq t - \delta$ and for all $\lambda \geq \lambda_0$ we have

$$\left\| \int_0^{t-\delta} e^{-e^{\lambda(t-s)}} T(s)x \, ds \right\| \leq \int_0^{t-\delta} e^{-e^{\lambda_0 \delta}} \|T(s)x\| \, ds.$$

Choose $\lambda_0 > \max(w, 0)$ such that the first term of the inequality is less than $\varepsilon/3$ for all $\lambda \geq \lambda_0$. Let $r > 0$. Then $1 - e^{-r} < r$. Therefore

$$\begin{aligned} \left\| \int_{t+\delta}^{\infty} (1 - e^{-e^{\lambda(t-s)}}) T(s)x \, ds \right\| &\leq \int_{t+\delta}^{\infty} e^{\lambda(t-s)} \|T(s)x\| \, ds \\ &\leq M \|x\| (\lambda - w)^{-1} e^{w(t+\delta) - \lambda \delta}. \end{aligned}$$

Chose $\lambda_1 \geq \lambda_0$ such that the third term of the inequality is less than $\varepsilon/3$ for all $\lambda > \lambda_1$. Hence, for every $x \in E$,

$$\int_0^t T(s)x \, ds = \lim_{\lambda \rightarrow \infty} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!} e^{n\lambda t} R(n\lambda, A)x.$$

Now let $x \in D(A)$. Then, by $R(\lambda, A)Ax = \lambda R(\lambda, A)x - x$, we obtain

$$\begin{aligned}
 T(t)x &= x + \int_0^t T(s)Ax \, ds \\
 &= x + \lim_{\lambda \rightarrow \infty} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!} e^{n\lambda t} [n\lambda R(n\lambda, A)x - x] \\
 &= x + \lim_{\lambda \rightarrow \infty} (e^{-e^{\lambda t}} - 1)x \\
 &\quad + \lim_{\lambda \rightarrow \infty} \lambda \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(n-1)!} e^{n\lambda t} R(n\lambda, A)x \\
 &= \lim_{\lambda \rightarrow \infty} \lambda \cdot \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(n-1)!} e^{n\lambda t} R(n\lambda, A)x. \quad \square
 \end{aligned}$$

Replacing A by $B = -s_0 + A$ and $T(t)$ by $e^{-s_0 t} T(t)$ one obtains for $x \in D(A)$, $s_0 \in \mathbf{C}$, and $r > 0$,

$$T(t)x = r e^{s_0 t} \lim_{\substack{k \rightarrow \infty \\ k \in \mathbf{N}}} k \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{(n-1)!} e^{nkr t} R(s_0 + nkr, A)x.$$

Hence the solutions of the initial value problem are completely determined by $R(\lambda, A)$, $\lambda \in \{s_0 + nr, n \geq n_0\}$.

For every $x \in E$ we have

$$\int_0^t e^{-s_0 r} T(r)x \, dr = \lim_{\substack{k \rightarrow \infty \\ k \in \mathbf{N}}} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n!} e^{nkr t} R(s_0 + nkr, A)x.$$

Using this equation and the fact that $R(\lambda, A)x$ is given as the Laplace transform of $T(t)x$ we get the following corollary.

COROLLARY. *Let A be the generator of a strongly continuous semigroup $(T(t))$ on a Banach space E and let Y be a closed subspace of E . Then the following statements are equivalent:*

- (i) *Y is an invariant subspace of $(T(t))$.*
- (ii) *There exists an $s_0 \in \mathbf{C}$, and $r \in \mathbf{R}$, and an $n_0 \in \mathbf{N}$ such that Y is an invariant subspace of $R(s_0 + nr, A)$ for every $n \geq n_0$.*

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20057