A CLOSED SURFACE OF GENUS ONE IN E^3 CANNOT CONTAIN SEVEN CIRCLES THROUGH EACH POINT

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ABSTRACT. There exists a closed surface of genus one in E^3 which contains six circles through each point, but any closed surface of genus one in E^3 cannot contain seven circles through each point.

1. Introduction. A sphere in E^3 is characterized as a closed surface which contains an infinite number of circles through each point. But we do not know a surface other than a sphere or a plane, which contains many circles through each point of it.

In 1980, Richard Blum [1] found a closed C^∞ surface of genus one which contains six circles through each point, and he gave a conjecture: A closed C^∞ surface in E^3 which contains seven circles through each point is a sphere.

We proved in [3] that a closed simply connected C^∞ surface in E^3 which contains three circles through each point is a sphere.

The purpose of this paper is to obtain the following theorem for a closed C^∞ surface of genus one.

THEOREM. *A closed C^∞ surface of genus one in E^3 cannot contain seven circles through each point.*

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2. Circles on a closed surface of genus one. Let M be a closed surface of genus one. Then M is topologically obtained from a square $ABCD$ by identifying $\overrightarrow{AB}$ with $\overrightarrow{DC}$ and $\overrightarrow{BC}$ with $\overrightarrow{AD}$.

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Let α and β be the closed curves on M corresponding to AB and BC , respectively. Then the homotopy classes $[\alpha]$ and $[\beta]$ defined by α and β are generators of the fundamental group $\pi_1(M)$ of M (see the diagram). Then the intersection numbers are given as follows:

$$\begin{aligned} \text{Int}(\alpha, \alpha) &= 0, & \text{Int}(\beta, \beta) &= 0, \\ \text{Int}(\alpha, \beta) &= 1 & \text{and} & \text{Int}(\beta, \alpha) = -1. \end{aligned}$$

Let γ be a closed curve on M . Then the homotopy class $[\gamma]$ defined by γ can be written as $[\gamma] = m[\alpha] + n[\beta]$ for some integers m and n .

The following facts are basic (see, for example, [4]).

FACT 1. γ is a simple closed curve if and only if m and n are relatively prime.

FACT 2. γ is knotted if and only if $|m| \geq 2$ and $|n| \geq 2$, or M is knotted and $n \neq 0$.

We see from these facts that

(*) if γ is a circle, then either $m = n = 0$ or $|m| = 1$ or $|n| = 1$.

REMARK. Since we disregard the orientation of curves, we identify γ with $-\gamma$.

3. Lemmas. Let M be a closed C^∞ surface of genus one in E^3 . Then we have the following lemmas for curves on M in view of (*):

LEMMA 1. *If two circles on M are homotopic and if they have only one point in common, then they are tangent to each other at the point.*

PROOF. Let c_1 and c_2 be two circles on M which belong to a homotopy class $m[\alpha] + n[\beta]$. Since the intersection number $\text{Int}(\ , \)$ is bilinear, $\text{Int}(c_1, c_2) = \text{Int}(m\alpha + n\beta, m\alpha + n\beta) = m^2 \text{Int}(\alpha, \alpha) + mn \text{Int}(\alpha, \beta) + nm \text{Int}(\beta, \alpha) + n^2 \text{Int}(\beta, \beta) = 0$.

Therefore c_1 and c_2 must be tangent to each other (cf. [4]).

LEMMA 2. *Let $c_1 \in m[\alpha] + [\beta]$ and $c_2 \in [\alpha] + n[\beta]$. If $mn \geq 4$ or $mn \leq -2$, then at least one of c_1 and c_2 cannot be a circle.*

PROOF. By the assumption, $\text{Int}(c_1, c_2) = mn - 1 \geq 3$ or ≤ -3 , so that c_1 and c_2 must have more than two points in common.

LEMMA 3. (1) *If $|n' - n| = 2$, then two circles $c_1 \in [\alpha] + n[\beta]$ and $c_2 \in [\alpha] + n'[\beta]$ have two points in common.*

(2) *If $|n' - n| \geq 3$, then two curves $c_1 \in [\alpha] + n[\beta]$ and $c_2 \in [\alpha] + n'[\beta]$ must have more than two points in common, so that at least one of them cannot be a circle.*

(3) *Similar results hold for two curves $c_1 \in m[\alpha] + [\beta]$ and $c_2 \in m'[\alpha] + [\beta]$.*

PROOF. (1) By the assumption, $\text{Int}(c_1, c_2) = n' - n = \pm 2$, so that c_1 and c_2 have two points in common.

(2) By the assumption, $\text{Int}(c_1, c_2) = n' - n \geq 3$ or ≤ -3 , so that c_1 and c_2 must have more than two points in common.

LEMMA 4. *If two circles $c_1 \in 0 \cdot [\alpha] + 0 \cdot [\beta]$ and c_2 have only one point in common, then they must be tangent to each other at the point.*

PROOF. $\text{Int}(c_1, c_2) = 0$, so that c_1 and c_2 must be tangent to each other at the point.

4. Proof of the theorem. Now we will prove our theorem by using the above lemmas and the following result:

PROPOSITION [3]. *Let M be a C^∞ surface in E^3 . Suppose that, through each point of M , there exist three circles of E^3 contained in M , any two of which are tangent to each other or have two points in common. Then M is (a part of) a sphere or a plane.*

Let p be an arbitrary point of M . Then, from Lemmas 2 and 3, we see that the maximal sets of homotopy classes which may contain circles through p simultaneously are

$$\{ @, [\alpha], m[\alpha] + [\beta], (m+1)[\alpha] + [\beta], (m+2)[\alpha] + [\beta] \}$$

for some integer m , or

$$\{ @, [\beta], [\alpha] + n[\beta], [\alpha] + (n+1)[\beta], [\alpha] + (n+2)[\beta] \}$$

for some integer n , where $@$ stands for $0 \cdot [\alpha] + 0 \cdot [\beta]$.

It follows from Lemmas 1–4 that each of the above sets is divided into three subsets with respect to the property that any two circles through p which belong to homotopy classes in one subset are tangent to each other or have two points in common:

$$\begin{aligned} & \{ @, [\alpha], m[\alpha] + [\beta], (m+1)[\alpha] + [\beta], (m+2)[\alpha] + [\beta] \} \\ &= \{ @, [\alpha] \} \cup \{ m[\alpha] + [\beta], (m+2)[\alpha] + [\beta] \} \cup \{ (m+1)[\alpha] + [\beta] \}, \\ & \{ @, [\beta], [\alpha] + n[\beta], [\alpha] + (n+1)[\beta], [\alpha] + (n+2)[\beta] \} \\ &= \{ @, [\beta] \} \cup \{ [\alpha] + n[\beta], [\alpha] + (n+2)[\beta] \} \cup \{ [\alpha] + (n+1)[\beta] \}. \end{aligned}$$

Suppose there exist seven circles through p . Then at least one subset must contain three circles and any two of them are tangent to each other or have two points in common. Since p is arbitrary, M must be a sphere or a plane by the Proposition. This contradicts the fact that M is of genus one. Q.E.D.

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