A LIL FOR OCCUPATION TIMES OF STABLE PROCESSES

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ABSTRACT. We prove a Strassen-type law of iterated logarithms for the occupation times of an R^d -valued ($d \geq 1$) stable process with the scaling property and positive density functions. An immediate application of our result is to obtain the asymptotic behavior of the occupation times of a path occupied in large spheres.

1. Introduction and main result. The purpose of this paper is to prove a Strassen-type law of iterated logarithms for the occupation times of a stable process. First, we review the result in Donsker and Varadhan [3] which is our motivation for study. Let $\{x(s), 0 \leq s < \infty\}$ be Brownian motion in R^d , $d \geq 3$ (so that the paths are transient), and let $\sigma(\omega, A)$ denote the total occupation time of a particular path $x(s, \omega)$ occupied in a Borel set $A \subset R^d$, i.e.,

$$\sigma(\omega, A) = \int_0^\infty \chi_A(x(s, \omega)) ds.$$

For each ω , $\sigma(\omega, \cdot)$ is a σ -finite measure on R^d . If we let M be the space of all σ -finite measures on R^d and impose the vague topology on M , then we have

DONSKER AND VARADHAN [3, THEOREM 1.1]. Define, for $t \geq 3$,

$$W_t(\omega, A) = \frac{\sigma(\omega, tA)}{t^2 \log \log t}.$$

Let $\Phi(\sigma)$ be continuous (in the vague topology) at the points of

$$\{\sigma \in M : I(\sigma) \leq 1\}.$$

Then, for almost all ω ,

$$\overline{\lim}_{t \rightarrow \infty} \{\Phi(W_t(\omega, \cdot))\} = \sup_{\sigma : I(\sigma) \leq 1} \Phi(\sigma).$$

In the above, $I(\sigma)$, the I -function of σ , is defined by

$$I(\sigma) = - \inf_{u \in U} \int_{R^d} \left(\frac{\frac{1}{2} \Delta u}{u} \right) (x) \sigma(dx),$$

where U is a suitable subspace of the domain of Δ .

We prove a result analogous to the above theorem of Donsker and Varadhan for general stable processes, including planar Brownian motion (recall that the

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latter is recurrent). We cannot obtain such a result by routine modifications of the procedure in [3], since some specific properties of transient Brownian paths play an indispensable role in that paper. We shall, instead of total occupation times, consider truncated occupation times:

$$\sigma(\omega, A, t) = \int_0^t \chi_A(x(s, \omega)) ds.$$

For a stable process $\{x(s), 0 \leq s < \infty\}$ in R^d ($d \geq 1$) with index α , we define

$$(1.1) \quad Z_t(\omega, A) = \frac{\sigma(\omega, tA, t^\alpha \log \log t)}{t^\alpha \log \log t}, \quad t \geq 3.$$

Now, for each ω , $Z_t(\omega, \cdot)$ is a probability measure (p.m.) on R^d . Let M now denote the space of all sub-p.m. on R^d , equipped with the vague topology too. For $\sigma \in M$, define

$$I(\sigma) = - \inf_{u \in \mathcal{D}^+} \int_{R^d} \left(\frac{Lu}{u} \right) (x) \sigma(dx)$$

where L is the infinitesimal generator of $\{x(s)\}$ and $\mathcal{D}^+$ is the space of positive functions in the domain of L (this definition follows the general definition of the I -function described in Donsker and Varadhan [1]). Our result is

THEOREM 1.1. *Let $\{x(s), 0 \leq s < \infty\}$ be an R^d -valued stable process with index α . Assume that $\{x(s)\}$ has the scaling property and that $x(s)$ has an everywhere positive (> 0) density. Let $\Phi(\sigma)$ be continuous at the points of $\mathcal{B} = \{\sigma \in M: I(\sigma) \leq 1\}$. Then, for almost all ω ,*

$$\varlimsup_{t \rightarrow \infty} \Phi(Z_t(\omega, \cdot)) = \sup_{\sigma: I(\sigma) \leq 1} \Phi(\sigma).$$

Note that our assumptions on $\{x(s)\}$ exclude nonsymmetric Cauchy processes and subordinators. However, we do not assume the transience of paths. An immediate application of Theorem 1.1, similar to that in [3], is to let $\Phi(\sigma) = \sigma(S_d(0, 1))$, where $s_d(0, 1)$ is the unit sphere of R_d , and thus to obtain the asymptotic behavior of the occupation times of the path occupied in large spheres (the case of small spheres [3, Theorem 1.2] is not considered in this paper). Reword it to indicate the asymptotic nature of the result, for large λ , Donsker and Varadhan [3] showed that the total occupation time of a transient Brownian path occupied in the sphere $S(0, \lambda)$ is a.s. (const.) $(\lambda^2 \log \log \lambda)$; while we show that the same effect happens when we consider the occupation time only up to $\lambda^2 \log \log \lambda$.

The proof of Theorem 1.1 follows the usual scheme. Let C_ω denote the set of limit points of $\{Z_t(\omega, \cdot)\}$ (as $t \rightarrow \infty$) in M . We prove that $\mathcal{B} = C_\omega$ a.s. In §2, we list some preliminaries and prove that $\mathcal{B} \supset C_\omega$, then we prove that $\mathcal{B} \subset C_\omega$ in §3.

The author is indebted to the referee for noticing that we should exclude the subordinators in order to apply the estimates of Donsker and Varadhan [2].

2. $C_\omega \subset \mathcal{B}$ a.s. Let $\{x(s), 0 \leq s < \infty\}$ be an R^d -valued ($d \geq 1$) stable process with index α . We refer to Taylor [4] for the definition and basic properties of such processes, we assume that $\{x(s)\}$ has the scaling property: $x(s)$ and $r^{-1/\alpha}x(rs)$ have the same distribution (abbrev. $x(s) \sim r^{-1/\alpha}x(rs)$) for any $r > 0$. We also assume that $x(s)$ has an everywhere positive (> 0) density. Except nonsymmetric Cauchy processes and subordinators, all other stable processes satisfy our assumptions.

Recall that $\{x(s)\}$ is transient if and only if $\alpha < d$, but we do not concern ourselves with this aspect in this paper. We may assume that every path $x(\omega, \cdot)$ is right continuous and has left limits everywhere.

Define, for $t \geq 3$, $\rho_t = (t/\log \log t)^{1/\alpha}$ and

$$\hat{L}_t(\omega, A) = \frac{1}{t} \int_0^t \chi_{\rho_t A}(x(s, \omega)) ds.$$

The functional $\hat{L}_t(\omega, \cdot)$ is the main concern in an important work on Donsker and Varadhan [2], in which “another law of iterated logarithms” was studied.

LEMMA 2.1. *For any $t \geq 3$ and any Borel sets $A_1, \dots, A_r$ in R^d , $\{Z_t(\omega, A_i)\}$ has the same joint distribution as that of $\{\hat{L}_t(\omega, A_i)\}$.*

PROOF. Write

$$\hat{L}_t(\omega, A) = \frac{1}{t} \int_0^t \chi_{tA} \left(\frac{tx(s)}{\rho_t} \right) ds,$$

and apply the scaling property $r^{1/\alpha}x(s) \sim x(rs)$ with $r = (\rho_t/t)^{-\alpha} = t^{\alpha-1}(\log \log t)$.

Thanks to Lemma 2.1, we are allowed to use nice estimates for the distribution of $\hat{L}_t(\omega, \cdot)$ appearing in [2]. In fact, our proof of Theorem 1.1 relies on transforming distributional estimates of $Z_t(\omega, \cdot)$ to those of $\hat{L}_t(\omega, \cdot)$. We should mention that the results in [2, §2] can be extended parallel to those R^d -valued stable processes satisfying our assumptions in Theorem 1.1.

PROPOSITION 2.2 $C_\omega \subset \mathcal{B}$. *a.s.*

To prove this, let $\beta \notin \mathcal{B}$ so that $I(\beta) > 1$. Since $I(\cdot)$ is lower semicontinuous and M is compact, we may find a weak neighborhood N of β such that $\theta = \inf_{\sigma \in \overline{N}} I(\sigma) > 1$. We may assume that N is of the form

$$(2.1) \quad N = \left\{ \sigma \in M : \left| \int \phi_i(x) \sigma(dx) - \int \phi_i(x) \beta(dx) \right| < \varepsilon, 1 \leq i \leq r \right\},$$

where $\varepsilon > 0$ and $\phi_i(x) \in C_0^\infty(R^d)$.

Choose k such that $0 < k < 1$ and $k(1 + \theta) > 2$. Set $t_n = \exp(n^k)$ and write $Z_n = Z_{t_n}$, $\hat{L}_n = \hat{L}_{t_n}$. We prove that

$$(2.2) \quad P(\omega: Z_n(\omega, \cdot) \in N \text{ i.o.}) = 0.$$

Note that

$$\begin{aligned} P(\omega: Z_n(\omega, \cdot) \in N) &= P\left(\omega: \left| \int \phi_i(x) Z_n(\omega, dx) - \int \phi_i(x) \beta(dx) \right| < \varepsilon, 1 \leq i \leq r\right) \\ &= P\left(\omega: \left| \int \phi_i(x) \hat{L}_n(\omega, dx) - \int \phi_i(x) \beta(dx) \right| < \varepsilon, 1 \leq i \leq r\right) \\ &= P(\omega: \hat{L}_n(\omega, \cdot) \in N) \\ &\leq P(\omega: \hat{L}_n(\omega, \cdot) \in \overline{N}), \end{aligned}$$

in the above, the second equality is due to Lemma 2.1. Then, we can use (2.30)–(2.32) in [2] to conclude that $\sum P(\omega: Z_n(\omega, \cdot) \in N) < \infty$. Hence, (2.2) is proved, by the Borel-Cantelli Lemma. Let N_1 be another weak neighborhood of β such that $N_1 \subset \overline{N}_1 \subset N$. We prove that: for each ω such that $Z_n(\omega, \cdot) \in N$ for only finitely

many n , $Z_t(\omega, \cdot) \notin N_1$ when t is sufficiently large. Assume the contrary; then there exists $\tau_j \uparrow \infty$ such that $Z_{\tau_j}(\omega, \cdot) \in N_1$, for all τ_j . Let $t_{n_j} : t_{n_j-1} \leq \tau_j \leq t_{n_j}$. Since $(t_{n-1}/t_n) \rightarrow 1$ as $n \rightarrow \infty$ (due to our choice that $0 < k < 1$), the above conclusion will contradict the observation that

$$\|Z_{t_2}(\omega, \cdot) - Z_{t_1}(\omega, \cdot)\| \rightarrow 0 \quad \text{as } t_2 > t_1 \rightarrow \infty \text{ and } (t_1/t_2) \rightarrow 1,$$

where $\|\cdot\|$ denotes the variation norm of measures.

Thus, we have proved that, for almost all ω , $Z_t(\omega, \cdot) \notin N_1$ when t is sufficiently large; the latter means that $\beta \notin C_\omega$. Therefore, Proposition 2.2 is proved.

3. $\beta \in C_\omega$ a.s. Now, we prove that each sub-p.m. β with $I(\beta) \leq 1$ is in C_ω . In view of [2, Lemma 2.16], it suffices to prove

THEOREM 3.1. *Let β be a sub-p.m. on R^d such that $I(\beta) < 1$ and such that β is supported on $|x| < a$ for some $a > 0$. Let G be a weak neighborhood of β ; define*

$$E_t = \{\omega : Z_t(\omega, \cdot) \in G\}.$$

Then, for almost all ω , $\omega \in E_t$ for a sequence of t 's increasing to ∞ .

To prove Theorem 3.1, we define an associated p.m. of $Z_t(\omega, \cdot)$ by

$$\bar{Z}_{t_1, t_2}(\omega, A) = \frac{1}{(t_2^\alpha \log \log t_2)(1 - (t_1/t_2))} \int_{(t_2^\alpha \log \log t_2)(t_1/t_2)}^{t_2^\alpha \log \log t_2} \chi_{t_2 A}(x(s)) ds, \quad 0 < t_1 < t_2.$$

LEMMA 3.2. *If $t_1, t_2 \rightarrow \infty$ and $(t_1/t_2) \rightarrow 0$, then*

$$\|\bar{Z}_{t_1, t_2}(\omega, \cdot) - Z_{t_2}(\omega, \cdot)\| \rightarrow 0.$$

PROOF. Break the integration interval $[0, t_2^\alpha \log \log t_2]$ of $Z_{t_2}(\omega, \cdot)$ into $[0, (t_2^\alpha \log \log t_2)(t_1/t_2)]$ and $[(t_2^\alpha \log \log t_2)(t_1/t_2), t_2^\alpha \log \log t_2]$. $\square$

Return to the proof of Theorem 3.1. Since $I(\beta) = \theta < 1$, we can choose $k > 1$ but such that $(2k-1)(\theta+1) < 2$. Set $t_n = \exp[n^k]$. Since $k > 1$, $(t_{n-1}/t_n) \rightarrow 0$ as $n \rightarrow \infty$. Let G_1 be another weak neighborhood of β such that G_1 is of the form (2.1) and such that $G_1 \subset \bar{G}_1 \subset G$. Write $r_n = t_n^\alpha \log \log t_n$, $e_n = t_{n-1}/t_n$, $\theta_n \doteq t_n^{\alpha-1} \log \log t_n$, $\bar{Z}_n = \bar{Z}_{t_{n-1}, t_n}$, etc. Fix $a' : a < a'$; define

$$E_n = \left\{ \omega : \bar{Z}_n(\omega, \cdot) \in G_1, \sup_{0 \leq s \leq r_n e_n} |x(s)| < a t_n, \sup_{0 \leq s \leq r_n} |x(s)| < a' t_n \right\}.$$

Because of Lemma 3.2, Theorem 3.1 will be proved once we can show that

$$(3.1) \quad P(E_n \text{ i.o.}) = 1.$$

Let

$$\mathcal{F}_n = \sigma\{x(s) : 0 \leq s \leq r_n e_n\}.$$

Since $r_{n+1}e_{n+1} > r_n$, $E_n \in \mathcal{F}_{n+1}$. Thus, by a conditional Borel-Cantelli Lemma ([2, Lemma 2.14], but it seems to exist in the literature already), (3.1) is implied by

$$(3.2) \quad \sum_n P(E_n | \mathcal{F}_n) = \infty \quad \text{a.s.}$$

By the Markov property, we have

$$P(E_n | \mathcal{F}_n) = \chi_{B_n}(\omega) P_{x(r_n e_n, \omega)} \left\{ \omega' : \tilde{Z}_n(\omega', \cdot) \in G_1, \sup_{0 \leq s \leq r_n(1-e_n)} |x(s, \omega')| < a' t_n \right\},$$

where $B_n = \{\omega : \sup_{1 \leq s \leq r_n e_n} |x(s, \omega)| < a t_n\}$, $\tilde{Z}_n(\omega, A)$ = the integral obtained by replacing $x(s)$ in $\bar{Z}_n(\omega, A)$ by $x(s - r_n e_n)$, and P_x denotes the p.m. corresponding to the process starting at x . Define

$$\rho_n = (t_n / \log \log t_n)^{1/\alpha}, \quad \text{and} \quad L'_n(\omega, A) = \frac{1}{t_n - t_{n-1}} \int_0^{t_n - t_{n-1}} \chi_{\rho_n A}(x(s)) ds.$$

Applying the scaling property $x(s) \sim \theta_n^{-1/\alpha} x(\theta_n s)$, we see that

$$(3.3) \quad \inf_{|x| \leq a \rho_n} P_x \left\{ L'_n(\omega, \cdot) \in G_1, \sup_{0 \leq s \leq t_n - t_{n-1}} |x(s)| < a' \rho_n \right\}$$

is equal to

$$(3.4) \quad \inf_{|x| \leq a t_n} P_x \left\{ \tilde{Z}_n(\omega, \cdot) \in G_1, \sup_{0 \leq s \leq r_n(1-e_n)} |x(s)| < a' t_n \right\}.$$

Since (3.3) is the one appearing in [2, (2.50)], we can use [2, (2.50)–(2.53)] to obtain that

$$(3.5) \quad \sum_n P(E_n | \mathcal{F}_n) \geq \sum_n \frac{\chi_{B_n}(\omega)}{n^{\theta'}}, \quad \theta' < 1.$$

We may use again the above scaling property and the arguments below [2, (2.54)] to conclude that the last sum in (3.5) is ∞ a.s. Therefore (3.1) is proved.

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