METRIZATION OF ADDITIVE κ -METRIC SPACES

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ABSTRACT. We prove that an additive κ -metric space X is metrizable if X is one of the following: (1) X is pseudocompact, (2) X is a wM -space, (3) X is a paracompact β -space, especially stratifiable.

Introduction. Ščepin [16, 17] introduced the notions of κ -metrizable and capacity as a generalization of metric spaces and locally compact groups, and proved that κ -metrizable is productive [18]. Dranišnikov [6] defined the additivity of κ -metric space and asserted that if an additive κ -metric space is compact, it is metrizable. We shall remark that the compactness is essential. Indeed, the Sorgenfrey line is an additive κ -metric space [19], but not metrizable. In this paper, we prove the following main theorem concerning metrization problems.

MAIN THEOREM. *An additive κ -metric space (X, d) , is metrizable if X is one of the following:*

- (1) X is pseudocompact.
- (2) X is a wM -space, especially an M -space.
- (3) X is a paracompact β -space, especially stratifiable.
- (4) d satisfies the (a, b) -inequality.

Let $\text{Rc}(X)$ be the collection of regular closed subsets of X . (X, d) is said to be an additive κ -metric space if there exists a mapping $d: X \times \text{Rc}(X) \rightarrow [0, \infty)$ satisfying the following:

- (K1) $d(x, C) = 0 \Leftrightarrow x \in C$.
- (K2) $C \subset D \Rightarrow d(x, C) \geq d(x, D)$ for $x \in X$.
- (K3) $d(x, C)$ is continuous with respect to x for every $C \in \text{Rc}(X)$.
- (AK4) $d(x, \text{cl}(\bigcup C_a; a \in A)) = \inf\{d(x, C_a); a \in A\}$.

d is called an additive κ -metric on X .

Throughout this paper, we mean by a space a Tychonoff space. We denote by $\mathfrak{N}$ the set of positive integers, and by " $y_n \rightarrow y$ " that the sequence $\{y_n; n \in \mathfrak{N}\}$ converges to y . For a given closed set F of an additive κ -metric space (X, d) we put $\text{Rc}(X \cdot F) = \{C; F \subset C \in \text{Rc}(X)\}$, $\text{Rci}(X, F) = \{C; F \subset \text{int } C \text{ and } C \in \text{Rc}(X)\}$, and $\text{Rci}(y) = \text{Rci}(X, \{y\})$. For a closed set F , we define

$$z(x, F) = \sup\{d(x, C); C \in \text{Rci}(X, F)\}$$

and put $d(F, D) = \inf\{d(x, D); y \in F\}$, $z(C, F) = \inf\{z(x, F); x \in C\}$, and $z(C, y) = \inf\{z(x, y); x \in C\}$. It is easy to see that (i) $z(x, C) \leq d(x, C)$ and (ii) $z(C, F) = \inf\{\inf\{z(x, y); x \in C\}; y \in F\}$ using Proposition 1.1(3). For metrization

Received by the editors September 25, 1985 and, in revised form, February 25, 1986.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54E20, 54E30, 54E35.

Key words and phrases. Metrization, additive κ -metric, stratifiable spaces, developable spaces, β -spaces.

problems, our definition of $z(x, F)$ seems to be more useful than Dranišnikov's: $z(x, F) = \sup\{d(x, C); C \in \text{Rc}(X, F)\}$.

1. Preliminaries.

PROPOSITION 1.1. *Let (X, d) be an additive κ -metric space. Then, using the notations defined above, we have*

- (1) $z(x, F) = 0 \Leftrightarrow x \in F$.
- (2) $U(x, a) = \text{int } S(x, a) \ni x$, where $S(x, a) = \{t; z(x, t) < 1/a\}$ when $a \in (0, \infty)$.
- (3) $z(x, F) = \inf\{z(x, y); y \in F\}$.
- (4) $x \notin F = \text{cl } F \Rightarrow z(U, F) > 0$ for some $U \in \text{Rci}(x)$.
- (5) $y_n \rightarrow y \Leftrightarrow z(y, y_n) \rightarrow 0$.
- (6) X is first countable and $\{U(x, n); n \in \mathfrak{N}\}$ is an open base at x .

PROOF. (1) Evident. (6) From (2) and (5).

(2) Since $z(x, y) \geq 1/a$ for every $y \notin S(x, a)$, there exists $C(y) \in \text{Rci}(y)$ with $d(x, C(y)) > 1/2a$. Put $C = \text{cl}(\bigcup\{C(y); y \notin S(x, a)\})$. Then $d(x, C) \geq 1/2a$ by (AK4), so $x \in X - C \subset S(x, a)$.

(3) Without loss of generality, we may assume $z(x, F) = b > 0$. Now suppose that $z(x, y) = e < b$ for some $y \in F$. By the definition of $z(x, y)$ and $z(x, F)$, $d(x, C(y)) \leq e$ for every $C(y) \in \text{Rci}(y)$ and there exists $C \in \text{Rci}(X; F)$ with $d(x, C) > e$. We take $C(y) \subset \text{cl } V(y) \in \text{Rci}(y)$ such that $V(y)$ is an open set containing y and $\text{cl } V(y) \subset C$. Then $d(x, C(y)) \leq e$, so $d(x, C) \leq e$, a contradiction. Now suppose that $z(x, y) \geq e > b = e - 2c$ for every $y \in F$ and some $e > 0$. For every $y \in F$, there exists $C(y) \in \text{Rci}(y)$ with $d(x, C(y)) \geq b + c$. Let $C = \text{cl}(\bigcup\{C(y); y \in F\})$. Then $b > d(x, C) = \inf\{d(x, C(y); y \in F\} > b + c$, a contradiction. Thus we have $z(x, F) = \inf\{z(x, y); y \in F\}$.

(4) Let $z(x, F) = 2b > 0$ and take $C \in \text{Rci}(X, F)$ with $d(x, C) > b$. By (K3) there exists $U \in \text{Rci}(x)$ with $d(t, C) > b$ for $t \in U$, and hence $z(t, F) \geq b$ for $t \in U$, so $z(U, F) > 0$.

(5) $(\Rightarrow)$ Suppose that there is $e > 0$ and $z(y, y_{i(n)}) > e$ for $\{y_{i(n)}; n \in \mathfrak{N}\} \subset \{y_n; n \in \mathfrak{N}\}$. For each $y_{i(n)}$, take $C_{i(n)} \in \text{Rci}(y_{i(n)})$ with $d(y, C_{i(n)}) > e$. Put $A = \text{cl}(\bigcup\{C_{i(n)}; n \in \mathfrak{N}\})$. Then $y \in A$ and $0 = d(y, A) > e$ using (AK4), a contradiction, so $z(y, y_n) \rightarrow 0$.

$(\Leftarrow)$ If $y_n \nrightarrow y$, there exists a subsequence $\{y_{i(n)}; n \in \mathfrak{N}\}$ with $y \notin F = \text{cl}(\bigcup\{y_{i(n)}; n \in \mathfrak{N}\})$, but by (4) this contradicts $z(y, y_n) \rightarrow 0$.

In the study of metrization of additive κ -metric spaces, various kinds of spaces X in Lemma 1.2(8) below will play a role. These spaces can be characterized by a generalized metric function $G: X \times \mathfrak{N} \rightarrow O(X)$, where $O(X)$ is the collection of open sets of X (for details, see [12]). The following results 1.2(1)–1.2(8) are well known, and these will be used as lemmas subsequently.

LEMMA 1.2. *Let X be a space.*

- (1) X is a β -space iff a sequence $\{x_n\}$ has a cluster point x when $x \in G(x_n, n)$, $n \in \mathfrak{N}$ [11].
- (2) X is $w\Delta$ iff a sequence $\{y_n\}$ has a cluster point x when $\{x, y_n\} \subset G(x_n, n)$, $n \in \mathfrak{N}$ [3].
- (3) X is semistratifiable iff $x_n \rightarrow x$ when $x \in G(x_n, n)$, $n \in \mathfrak{N}$ [8].
- (4) X is semimetric iff X is first countable and semistratifiable [8].

(5) X is developable iff $y_n \rightarrow x$ when $\{x, y_n\} \subset G(x_n, n)$, $n \in \mathfrak{N}$ [9], or iff X is $w\Delta$ and semistratifiable [5, 8].

(6) X is stratifiable iff for $p \notin A = \text{cl } A$, there exists $n \in \mathfrak{N}$ with $p \notin \text{cl}(\bigcup\{G(x, n); x \in A\})$ [4].

(7) X is metrizable iff X is paracompact and developable [1] or iff X is wM and semistratifiable [14]; further, iff (i) $\{G(x, n); n \in \mathfrak{N}\}$ is a base at x and (ii) for a given $G(x, n)$ there exists $m = m(x, n)$ such that $G(x, m) \cap G(y, m) \neq \emptyset$ implies $G(y, m) \subset G(x, n)$ (Frink's theorem [7]).

(8) The following implications are well known.

$$\begin{array}{ccccc} \text{developable} & \Rightarrow & \text{semimetric} & & \text{paracompact} \\ & \searrow & \downarrow & & \uparrow \\ M \Rightarrow wM \Rightarrow w\Delta \Rightarrow \beta \Leftarrow \text{semistratifiable} \Leftarrow \text{stratifiable} \end{array}$$

For definitions of M -spaces and wM -spaces, see [15 and 16] respectively. We note that (1) if X is stratifiable, then X is paracompact and perfectly normal [12], and (2) Heath showed an example of a paracompact semimetric space which is not stratifiable [10].

2. Results.

THEOREM 2.1. For any additive κ -metric space (X, d) , we have the following implications: (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) $\Leftrightarrow$ (5) $\Rightarrow$ (6), and (3) $\Rightarrow$ (7).

(1) X is pseudocompact.

(2) $x \notin C \in \text{Rc}(X) \Rightarrow d(C, D) > 0$ for some $D \in \text{Rci}(x)$.

(3) $x \notin C \in \text{Rc}(X) \Rightarrow z(C, E) > 0$ for some $E \in \text{Rci}(x)$.

(4) $x \notin C \in \text{Rc}(X) \Rightarrow z(C, x) > 0$.

(5) $z(y_n, x) \rightarrow 0 \Rightarrow y_n \rightarrow x$.

(6) X is developable.

(7) X is stratifiable.

PROOF. (1) $\Rightarrow$ (2) Take $D \in \text{Rci}(x)$ with $C \cap D = \emptyset$. Since $d(y, D) > 0$ for every $y \in C$ and C is pseudocompact, we have $d(C, D) > 0$.

(2) $\Rightarrow$ (3) Take n with $E = \text{cl } S(x, n) \subset \text{int } D$. By the definition of $z(y, E)$ we have $z(y, E) \geq d(y, D)$, so $z(C, E) \geq d(C, D) > 0$.

(3) $\Rightarrow$ (2) From (3) and $d(y, K) \geq z(y, K)$ for $K \in \text{Rc}(X)$.

(3) $\Rightarrow$ (4) Evident.

(4) $\Rightarrow$ (5) Suppose that there exists $A = \{Y_{i(n)}; n \in \mathfrak{N}\}$ with $x \notin F = \text{cl } A$. Take $C \in \text{Rci}(X, F)$ with $x \notin C$. Then $z(x, C) > 0$ by (4), a contradiction.

(5) $\Rightarrow$ (4) If $z(C, x) = 0$, $z(y_n, x) \rightarrow 0$ for some $y_n \in C$, $n \in \mathfrak{N}$. By (5), $y_n \rightarrow x$. Since C is closed, $y \in C$, a contradiction.

(5) $\Rightarrow$ (6) We replace $G(x, n)$ in Lemma 1.2(4) by $U(x, n)$. If $\{x, y_n\} \subset U(x_n, n)$, then $x \in U(x_n, n)$, and hence $x_n \rightarrow x$ by (5), so X is semistratifiable by Lemma 1.2(3). Thus by Lemma 1.2(5) it suffices to show that X is $w\Delta$, that is, the sequence $\{y_n\}$ has a cluster point. If $F = \{y_n; n \in \mathfrak{N}\}$ is a discrete closed set, here we can assume that $x \notin F$, and hence $z(U, F) > 0$ for some $U \in \text{Rci}(x)$ by Proposition 1.1(4). By Proposition 1.1(3) this contradicts $y_n \in U(x_n, n)$, $n \in \mathfrak{N}$.

(3) $\rightarrow$ (7) Suppose that $x \notin C \in \text{Rci}(X, A)$. We take n such that $\text{cl } S(x, n) \subset D$ and $z(C, D) > 1/n$ where D is the set in (3). For $y \in C$, $z(y, t) > 1/n$ implies

$t \notin U(x, n)$, i.e., $S(y, n) \cap U(x, n) = \emptyset$, and hence

$$U(x, n) \cap \text{cl}(\bigcup \{S(y, n); y \in A\}) = \emptyset,$$

so $x \notin \text{cl}(\bigcup \{U(y, n); y \in A\})$. By Lemma 1.2(6) X is stratifiable.

We note that it is easy to see that for any metric space (X, ρ) , the function d , defined by $d(x, C) = \rho(x, C)$ for $C \in \text{Rc}(X)$, is an additive κ -metric satisfying Theorem 2.1(2).

THEOREM 2.2. *If an additive κ -metric space (X, d) is a β -space, then X is developable.*

PROOF. Let us put $H(x, n) = U(x, n) \cap G(x, n)$ and $x \in H(x_n, n)$, $n \in \mathfrak{N}$, where $G(x, n)$ is the open set in Lemma 1.2(1) with $G(x, n) \supset G(x, n+1)$ for every $n \in \mathfrak{N}$. Since X is a β -space and $x \in G(x_n, n)$, the sequence $\{x_n\}$ has a cluster point p . If $x \neq p$, then by Proposition 1.1(4) there exists $U \in \text{Rci}(p)$ with $z(U, x) > 0$. This contradicts $x \in U(x_n, n)$, $n \in \mathfrak{N}$. Thus we have $x = p$. Next we shall show using Lemma 1.2(3) that X is semistratifiable. If $x_n \rightarrow x$, there exists $U \in \text{Rci}(x)$ such that $X - U \supset F = \{x_{i(n)}; n \in \mathfrak{N}\}$. Define

$$t_{i(n)+k} = x_{i(n+1)} \quad \text{for } 0 \leq k < i(n+1) - i(n).$$

Since $H(t_{i(n)+k}, i(n)+k) = H(x_{i(n+1)}, i(n)+k) \supset H(x_{i(n+1)}, i(n+1)) \ni x$, we have $x \in H(t_n, n)$. But the sequence $\{t_n\}$ has no cluster point because $t_n \in F$, $n \in \mathfrak{N}$, a contradiction, i.e., X is semistratifiable. The proof that X is $w\Delta$ follows from the same method in the proof of the latter part of (5) $\Rightarrow$ (6) in Proposition 2.1. Thus X is developable by Lemma 1.2(5).

We consider the following condition for an additive κ -metric space (X, d) .

(a, b)-inequality. For any $p, q \in X$, and for any $C(q) \in \text{Rci}(q)$, there exist positive real numbers a and b such that the following inequality holds for every $t \in X$:

$$d(p, C(q)) \leq a \cdot \sup\{d(t, C(p)); C(p) \in \text{Rci}(p)\} + b \cdot d(t, C(q)).$$

THEOREM 2.3. *If an additive κ -metric space (X, d) satisfies the (a, b)-inequality, then we have:*

- (1) $z(p, q) \leq a \cdot z(t, p) + b \cdot z(t, q)$.
- (2) $z(p, q) \leq a \cdot z(q, p)$, $a \geq 1$ and $b \geq 1$.
- (3) $z(p, q) \leq a^2 \cdot z(p, t) + ab \cdot z(q, t)$.
- (4) $z(p, q) \leq a \cdot z(t, p) + ab \cdot z(q, t)$.
- (5) $z(p, q) \leq a^2 \cdot z(p, t) + b \cdot z(t, q)$.
- (6) Let $n(a^2 + b) < m$ for $m, n \in \mathfrak{N}$. Then $S(p, m) \ni x \Rightarrow S(x, m) \subset S(p, n)$ and $U(p, m) \ni x \Rightarrow U(x, m) \subset U(p, n)$.
- (7) Let $n(a^4 + a^3b + b) < m$ for $m, n \in \mathfrak{N}$. Then $S(x, m) \cap S(p, m) \neq \emptyset \Rightarrow S(x, m) \subset S(p, n)$ and $U(x, m) \cap U(p, m) \neq \emptyset \Rightarrow U(x, m) \subset U(p, n)$.

PROOF. (1) Since $d(t, C(q)) \leq z(t, q)$ and $\{\text{cl } U(x, n); n \in \mathfrak{N}\}$ is a base at p , we have $d(p, C(q)) \leq a \cdot \lim d(t, \text{cl } U(p, n)) + b \cdot z(t, q) \leq a \cdot z(t, p) + b \cdot z(x, q)$ and hence $z(p, q) \leq a \cdot z(t, p) + b \cdot z(t, q)$. (2)–(7) follow from (1) by direct calculations.

We note that if an additive κ -metric space (X, d) satisfies the (1, 1)-inequality, then $z(x, y)$ is a distance function on X , and hence X is metrizable. For, by Theorem 2.3(2), $z(p, q) = z(q, p)$ for $p, q \in X$, and by Theorem 2.3(5), $z(x, y)$ satisfies the triangle inequality. By simple calculations, we have that $z(x, y)$ is continuous in both variable x and y , so $z(x, y)$ is a distance function on X .

3. Proof of the main theorem. (1) X being pseudocompact, X is developable and stratifiable by Theorem 2.1, and hence paracompact by Lemma 1.2(8), so X is metrizable by Lemma 1.2(7).

(2) Since X is wM , it is a β -space by Lemma 1.2(8). On the other hand, X is developable by Theorem 2.2, and hence semimetric by Lemma 1.2(8), so X is metrizable by Lemma 1.2(7).

(3) From Theorem 2.2 and Lemma 1.2(7).

(4) From Theorem 2.3(7) and Lemma 1.2(7).

4. Examples. 4.1. The additivity of κ -metrics is not productive. Let X be the product of uncountably many copies of $\mathfrak{N}$. If X has an additive κ -metric, then X is first countable by Proposition 1.1(6), but it is well known that X is not first countable, a contradiction.

4.2. For any infinite discrete space X , there exists an additive metric d such that (i) d satisfies neither Theorem 2.1(5) nor the (a, b) -inequality, and (ii) d is regular in the sense of Ščepin [18]. Let $X \supset \{a_n; n \in N\} \cup \{p\}$. Then a desired κ -metric d is defined as follows: $d(x, y) = 0$ if $x = y$, $d(a_n, p) = 1/n$ ($n \in \mathfrak{N}$), and $d(x, y) = 1$ otherwise.

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