A CONTINUUM X WHICH IS A RETRACT OF $C(X)$ BUT NOT OF 2^X

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ABSTRACT. We give an example of a continuum X which is a retract of $C(X)$ but not of 2^X .

1. Introduction. A *continuum* is a compact connected metric space. The *hyperspaces* of a continuum X are the spaces $2^X = \{A \in X: A \text{ is closed and non-empty}\}$ and $C(X) = \{A \in 2^X: A \text{ is connected}\}$, each with the Hausdorff metric. We identify X with the subspace $\{\{x\} \in 2^X: x \in X\}$ of 2^X (and $C(X)$). If p is a point of X , we define $C_p(X) = \{A \in C(X): p \in A\}$. A *map* is a continuous function. We say that a map $\lambda: X \times X \rightarrow X$ is a *mean* if $\lambda(x, y) = \lambda(y, x)$ for all $x, y \in X$ and $\lambda(x, x) = x$ for all $x \in X$.

It is known [4, Theorem 6.4] that if X is a locally connected continuum, then the following assertions are equivalent:

- (a) X is an absolute retract.
- (b) X is a retract of $C(X)$.
- (c) X is a retract of 2^X .

This result has motivated the question asked by S. B. Nadler, Jr. [4, question 6.28], of whether for every continuum, (b) and (c) are equivalent. The aim of this paper is to show a continuum X for which (b) holds while (c) does not hold (in fact X does not admit a mean). Other interesting examples concerning retractions between hyperspaces can be found in [1, 2, and 3].

I express my thanks to Professor D. Curtis for his suggestions to simplify the example and its construction.

2. Construction of the example X . Consider the triod $T = [-1, 1] \times \{0\} \cup \{0\} \times [0, 1] \subset \mathbf{R}^2$, the convergent sequence $J = \{1/n: n = 1, 2, \dots\} \cup \{0\} \subset \mathbf{R}$ and the cone $\text{Cone}(T \times J) = \{(s, t, j): 0 \leq s \leq 1, t \in T, j \in J\}$ with the cone point $p = (0, t, j)$.

Let $\mathcal{T} \subset 2^T$ denote the collection of all three-element subsets $\{x, y, z\}$ of T such that one of the points x, y , or z is between (i.e. separates) the other two points in T . Use the natural convex structure on T to define $ux + vy$ in T , for all $x, y \in T$, $u, v \geq 0$ with $u + v = 1$. For each $\sigma = \{x, y, z\} \in \mathcal{T}$ with z between x and y , define the triod $T_\sigma \subset \text{Cone}(T \times \{0\})$ as follows:

$$T_\sigma = \{(s, z, 0): 0 \leq s \leq 1/2\} \cup \{(s, (2s-1)x + (2-2s)z, 0): 1/2 \leq s \leq 1\} \\ \cup \{(s, (2s-1)y + (2-2s)z, 0): 1/2 \leq s \leq 1\}.$$

Let $\{\sigma_n: n = 1, 2, \dots\}$ be a dense sequence in $\mathcal{T}$, and for each n define the triod $T_n \subset \text{Cone}(T \times \{1/n\})$ by $T_n = \{(s, t, 1/n): (s, t, 0) \in T_{\sigma_n}\}$. Then define

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$X \subset \text{Cone}(T \times J)$ by $X = \bigcup \{T_n : n = 1, 2, \dots\} \cup \text{Cone}(T \times \{0\})$. (Of course, X could be imbedded in $\mathbf{R}^3$, but this would unnecessarily complicate its description.)

3. X does not admit a mean. Suppose there exists a mean $\lambda: X \times X \rightarrow X$. Let d be the path-length metric on the triod T . Choose $\varepsilon > 0$ such that, for every pair $(s, t, j), (s, t', j)$ in X with $s \geq 1/2$ and $d(t, t') \leq 2\varepsilon$, $\lambda((s, t, j), (s, t', j)) \neq p$, the cone point. For convenience, identify T with the subset $\{1\} \times T \times \{0\}$ of $\text{Cone}(T \times \{0\})$. Thus $T \subset X$.

CLAIM. For each $x, y \in T$ with $d(x, y) \leq \varepsilon$, $\lambda(x, y) \in \{x, y\}$. Assuming $x \neq y$, consider any $z \in T$ between x and y . Choose a subsequence $\{\sigma_{n_i}\}$ such that $\sigma_{n_i} \rightarrow \sigma = \{x, y, z\}$. Then $T_{n_i} \rightarrow T_\sigma$. Let e_{n_i}, f_{n_i} denote the endpoints of T_{n_i} different from p , labeled so that $e_{n_i} \rightarrow x$ and $f_{n_i} \rightarrow y$. By the choice of ε , we must have $\lambda(e_{n_i}, f_{n_i}) \in T_{n_i}$ for all n_i sufficiently large. Thus $\lambda(x, y) \in T_\sigma$. Since this holds for any $\sigma = \{x, y, z\}$ with z between x and y , it follows that $\lambda(x, y) \in \{x, y\}$.

Now consider the points $(0, 0), (\pm\varepsilon, 0), (0, \varepsilon)$ of T . We may suppose by the claim that $\lambda((0, 0), (\varepsilon, 0)) = (\varepsilon, 0)$. By continuity of λ , this implies that $\lambda((0, \varepsilon), (0, 0)) = (0, 0)$, and that $\lambda((-\varepsilon, 0), (0, 0)) = (0, 0)$. But the latter implies that $\lambda((0, 0), (0, \varepsilon)) = (0, \varepsilon)$, contradicting the hypothesis that λ is a mean.

4. $C(X)$ retracts to X . Let $l: X \rightarrow [0, 1]$ be the projection map defined by $l(s, t, j) = s$. Let $\mathcal{L} \subset C(\text{Cone}(T \times \{0\}))$ be the closed subhyperspace defined by $\mathcal{L} = \{M : M = \lim M_i, \text{ for some sequence } \{M_i\} \text{ in } C(X) \text{ such that } M_i \subset T_{n_i}, \text{ with } n_i \rightarrow \infty\}$. Then for each $M \in \mathcal{L}$, there exists a *unique* $x \in M$ such that $l(x) = \inf l(M)$.

Let $r: C(\text{Cone}(T \times \{0\})) \rightarrow \text{Cone}(T \times \{0\})$ be any retraction satisfying the conditions:

- (1) $r(M) = p$ if $p \in M$; and
- (2) for each $M \in \mathcal{L}$, $r(M) \in M$ is such that $l(r(M)) = \inf l(M)$.

Since $\text{Cone}(T \times \{0\})$ is an AR, such a retraction may be constructed by first defining $r|_{\text{Cone}(T \times \{0\})} = \text{id}$, then defining r over $C_p(\text{Cone}(T \times \{0\})) \cup \mathcal{L}$ by conditions (1) and (2), and finally, extending r over the entire hyperspace $C(\text{Cone}(T \times \{0\}))$.

Then the desired retraction $r: C(X) \rightarrow X$ is obtained as the extension of the above r satisfying the conditions:

- (1*) $r(M) = p$ if $p \in M$; and
- (2*) for each $M \in C(X)$ with $M \subset T_n$ for some n , $r(M) \in M$ is such that $l(r(M)) = \inf l(M)$.

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