COMPLETION OF FACTOR ALGEBRAS OF IDEALS

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ABSTRACT. Let $\mathfrak{I}$ be a κ -complete ideal over κ . The structure of the completion of the Boolean algebra $\wp(\kappa)/\mathfrak{I}$ is investigated with respect to properties of the ideal $\mathfrak{I}$ and the cardinal κ . It is shown that under certain conditions $\text{Comp}(\wp(\kappa)/\mathfrak{I})$ is isomorphic to a collapse algebra.

1. Introduction. In [BV] it is proven that $\text{Comp}(\wp(\kappa)/\mathfrak{I}_\kappa)$ and $\text{Col}(\omega, \kappa^+)$ are isomorphic if $2^\kappa = \kappa^+$ and κ is a regular uncountable cardinal (and where $\mathfrak{I}_\kappa$ is the ideal of all subsets of κ of size $< \kappa$). In this paper we extend this result to:

THEOREM 1. *Let κ be a regular uncountable cardinal such that $2^\kappa = \kappa^+$. Let $\mathfrak{I}$ be a κ -complete nowhere precipitous ideal over κ . Then $\text{Comp}(\wp(\kappa)/\mathfrak{I})$ and $\text{Col}(\omega, \kappa^+)$ are isomorphic.*

COROLLARY 2. *Assume $V = L$. For every regular uncountable cardinal κ and every κ -complete ideal $\mathfrak{I}$ over κ , $\text{Comp}(\wp(\kappa)/\mathfrak{I})$ and $\text{Col}(\omega, \kappa^+)$ are isomorphic.*

THEOREM 3. *Let κ be a singular cardinal. Assume $2^\kappa = \kappa^+$ and $2^{cf(\kappa)} = cf(\kappa)^+$.*

- (1) *If $cf(\kappa) = \omega$, then $\text{Comp}(\wp(\kappa)/\mathfrak{I})$ and $\text{Col}(\omega_1, \kappa^+)$ are isomorphic.*
- (2) *If $cf(\kappa) > \omega$, then $\text{Comp}(\wp(\kappa)/\mathfrak{I}_\kappa)$ and $\text{Col}(\omega, \kappa^+)$ are isomorphic.*

REMARK. "If $V = L$, then $\text{Comp}(\wp(\omega_1)/\mathfrak{I})$ and $\text{Col}(\omega, \omega_2)$ are isomorphic" is proven independently using a different method in [BTW].

REMARK. Let $\lambda = \mu^+ \leq \kappa$. Let $\mathfrak{I}$ be a λ -complete and not λ^+ -complete ideal over κ . Then $\wp(\kappa)/\mathfrak{I}$ as a forcing notion collapses either λ^+ to λ , or λ to μ (see [F]). If $\lambda = \kappa$, the latter happens (see [BTW]).

2. Notation and definitions. Lowercase Greek letters are reserved for ordinals. Let κ, τ, μ be cardinals.

Let X, Y be sets. $\wp(X)$ is the set of all subsets of X . XY is the set of all functions from X into Y ,

$$[X]^{<\lambda} = \{Y \subseteq X : |Y| < \lambda\}, \quad [X]^\lambda = \{Y \subseteq X : |Y| = \lambda\},$$

$$\kappa^{<\lambda} = \sum \{\kappa^\gamma : \gamma < \lambda \text{ \& } \gamma \text{ is a cardinal}\}, \quad {}^{<\lambda}\kappa = \bigcup \{\kappa^\alpha : \alpha \in \lambda\}.$$

$\text{Col}(\lambda, \kappa)$ is the Boolean completion of the set of all functions f with $\text{dom}(f) \in [\lambda]^{<\lambda}$ and $\text{rng}(f) \subseteq \kappa$ ordered by the inverse inclusion. (In generic extensions obtained via $\text{Col}(\lambda, \kappa)$, the cardinal λ is preserved and the cardinal κ becomes an ordinal of size λ .)

Received by the editors March 10, 1986.

1980 *Mathematics Subject Classification* (1985 Revision). Primary 06E05, 03G05; Secondary 03E05, 04A20.

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0002-9939/87 \$1.00 + \$.25 per page

If $X, Y \subseteq \wp(\kappa)$, then $X \subseteq Y$ iff $(\forall y \in Y)(\exists x \in X)(x \subseteq y)$; we say that X *refines* Y , or that X is a *refinement* of Y .

If B is a Boolean algebra, then 1_B is its greatest element, while 0_B is its least element. $B^+ = B - \{0_B\}$. If $b \in B$, then $B|b = \{c \in B : c \leq b\}$. $\text{Comp}(B)$ denotes the Boolean completion of B (it can be defined as the algebra of regular open sets of the Stone space of B and is unique up to isomorphism).

Let $b \in B^+$. $P \subseteq B^+$ is a *partition of b* iff P is a maximal disjoint subfamily of elements of $(B|b)^+$. If P_1, P_2 are partitions of b , then

$$P_1 \leq P_2 \quad \text{iff} \quad (\forall c \in P_1)(\exists d \in P_2)(c \leq d);$$

we say P_1 *refines* P_2 , or that P_1 is a *refinement* of P_2 . $\langle P_\alpha : \alpha < \lambda \rangle$ is a *descending sequence of partitions of b* iff $P_\alpha \leq P_\beta$ whenever $\beta < \alpha < \lambda$.

$\mathfrak{I}$ is a λ -complete ideal over κ iff $\mathfrak{I} \subseteq \wp(\kappa)$, $\mathfrak{I} \neq \wp(\kappa)$, $\mathfrak{I}$ is closed under unions of size $< \lambda$ and under the subset operation, and $\mathfrak{I}$ contains all singletons of κ . $\mathfrak{I}^+ = \wp(\kappa) - \mathfrak{I}$. $\mathfrak{I}^* = \{X \subseteq \kappa : (\kappa - X) \in \mathfrak{I}\}$, which is the dual filter to $\mathfrak{I}$.

For an $A \in \mathfrak{I}^+$ define $\mathfrak{I}|A = \{X \subseteq \kappa : (X \cap A) \in \mathfrak{I}\}$ (which also is a λ -complete ideal over κ and $\mathfrak{I} \subseteq \mathfrak{I}|A$).

$\mathfrak{I}_\kappa = \{X \subseteq \kappa : |X| < \kappa\}$ is the *Fréchet ideal* over κ .

Let $X, Y \in \mathfrak{I}$. Then $X \subseteq^* Y$ iff $(X - Y) \in \mathfrak{I}$. $P \subseteq \wp(\kappa)$ is $\mathfrak{I}$ -disjoint iff $(\forall X \neq Y \in P)(X \cap Y \in \mathfrak{I})$. $P \subseteq \wp(\kappa)$ is an *almost disjoint family* iff P is $\mathfrak{I}_\kappa$ -disjoint.

Let $S \in \mathfrak{I}^+$, then $P \subseteq \wp(S) \cap \mathfrak{I}^+$ is an $\mathfrak{I}$ -partition of S iff P is $\mathfrak{I}$ -disjoint and maximal. $\langle P_\alpha : \alpha \in \lambda \rangle$ is a *descending sequence of $\mathfrak{I}$ -partitions of S* iff for every $\alpha \in \lambda$, P_α is an $\mathfrak{I}$ -partition of S and $P_\alpha \subseteq P_\beta$ whenever $\alpha < \beta < \lambda$. Let $\langle P_n : n < \omega \rangle$ be a descending sequence of $\mathfrak{I}$ -partitions of S . Then a sequence $\langle X_n : n < \omega \rangle$ is a *path through $\langle P_n : n < \omega \rangle$* iff for every $n \in \omega$, $X_n \in P_n$ and also $X_{n+1} \subseteq X_n$.

$\mathfrak{I}$ is a *precipitous ideal* iff for every $S \in \mathfrak{I}^+$ and for every descending sequence $\langle P_n : n \in \omega \rangle$ of $\mathfrak{I}$ -partitions of S , there is a path through with a nonempty intersection. $\mathfrak{I}$ is *nowhere precipitous* iff for every $A \in \mathfrak{I}^+$, $\mathfrak{I}|A$ is not precipitous.

Let $X \subseteq \kappa$. Define

$$\begin{aligned} X/\mathfrak{I} &= \{Y \subseteq \kappa : (X - Y) \cup (Y - X) \in \mathfrak{I}\}, \quad \wp(\kappa)/\mathfrak{I} = \{X/\mathfrak{I} : X \subseteq \kappa\}, \\ X/\mathfrak{I} \wedge Y/\mathfrak{I} &= (X \cap Y)/\mathfrak{I}, \quad X/\mathfrak{I} \vee Y/\mathfrak{I} = (X \cup Y)/\mathfrak{I}, \quad -X/\mathfrak{I} = (\kappa - X)/\mathfrak{I}. \end{aligned}$$

Thus $\wp(\kappa)/\mathfrak{I}$, $\wedge$, and $\vee$ form a λ -complete Boolean algebra (for $\mathfrak{I}$ is λ -complete) with $\emptyset/\mathfrak{I} = \mathfrak{I}$ being its smallest element and $\kappa/\mathfrak{I}$ being its greatest element. For any $\xi < \lambda$,

$$\sum \{X_\alpha/\mathfrak{I} : \alpha < \xi\} = \left(\bigcup \{X_\alpha : \alpha < \xi\} \right) / \mathfrak{I}$$

and

$$\prod \{X_\alpha : \alpha < \xi\} = \left(\bigcap \{X_\alpha : \alpha < \xi\} \right) / \mathfrak{I}.$$

Let $\kappa \geq 2$ and $\lambda \geq \omega$. A Boolean algebra B is (λ, μ, κ) -*distributive* iff for every sequence $\langle P_\alpha : \alpha \in \lambda \rangle$ of partitions of 1_B of size $\leq \mu$, here is Q , a partition of 1_B , so that $(\forall \alpha \in \lambda)(\forall q \in Q)(|\{p \in P_\alpha : p \wedge q \neq 0_B\}| < \kappa)$. B is $(\lambda, \cdot, \kappa)$ -*distributive* iff the above holds without the restriction on the size of P_α 's. B is $(\lambda, \cdot, \kappa)$ -*nowhere distributive* iff for every $b \in B^+$, $B|b$ is not $(\lambda, \cdot, \kappa)$ -distributive.

Let P be a set partially ordered by $\leq$. $D \subseteq P$ is *dense* in P iff $(\forall p \in P)(\exists d \in D)(d \leq p)$. P is λ -*closed* iff for any descending sequence $\langle p_\alpha : \alpha < \beta \rangle$ of length $\beta < \lambda$ of elements of P , there is a $p \in P$ such that $(\forall \alpha < \beta)(p \leq p_\alpha)$. $d(P) = \min\{|D| : D \text{ is dense in } P\}$.

3. Preliminaries.

LEMMA 4. *Let λ be an infinite cardinal and κ be a cardinal ≥ 2 . A Boolean algebra B is $(\lambda, \cdot, \kappa)$ -nowhere distributive iff there is a sequence $\langle P_\alpha : \alpha < \lambda \rangle$ of partitions of 1_B such that $(\forall b \in B^+)(\exists \alpha \in \lambda)(|\{p \in P_\alpha : p \wedge b \neq 0_B\}| \geq \kappa)$.*

PROOF. See Lemma 1.11 in [BSV]. $\square$

LEMMA 5. *Let λ be an infinite cardinal and κ be a cardinal ≥ 2 . Let B be a complete $(\lambda, \cdot, \kappa)$ -nowhere distributive Boolean algebra containing a λ -closed dense subset. Let $d(B) = \kappa^{<\lambda}$. Then B is isomorphic to $\text{Col}(\lambda, \kappa)$.*

PROOF. See Theorem 1.15 and Corollary 1.16 in [BSV]. $\square$

LEMMA 6. *Let κ be an uncountable cardinal. Let $\mathfrak{I}$ be a countably complete ideal over κ . $\mathfrak{I}$ is nowhere precipitous iff there is a descending sequence of $\mathfrak{I}$ -partitions of κ with no path through whose intersection is nonempty.*

PROOF. Assume that $\mathfrak{I}$ is nowhere precipitous (the opposite direction is obvious). Note that if $A \in \mathfrak{I}^+$, then $(\mathfrak{I}|A)^+ \subseteq \mathfrak{I}^+$ and so if $C \in (\mathfrak{I}|A)^+$ and W is an $(\mathfrak{I}|A)$ -partition of C , $C \in \mathfrak{I}^+$ and W is an $\mathfrak{I}$ -partition of C as well. Since $\mathfrak{I}$ is nowhere precipitous, $\mathfrak{I}|A$ is not precipitous for any $A \in \mathfrak{I}^+$. I.e., there are $C \in (\mathfrak{I}|A)^+$ and a descending sequence of $(\mathfrak{I}|A)$ -partitions with no path through whose intersection is nonempty. Thus there is $C \in \mathfrak{I}^+$ so that $C \subseteq A$ and

- (*) there is a descending sequence $\langle W(C, n) : n < \omega \rangle$ of $\mathfrak{I}$ -partitions of C with no path through whose intersection is nonempty.

We have just shown that the set of all $C \in \mathfrak{I}^+$ for which (*) holds is dense in $(\mathfrak{I}^+, \subseteq)$. Therefore there is an $\mathfrak{I}$ -disjoint partition F of κ so that (*) holds for every $C \in F$. Now for every $n < \omega$ define $W_n = \bigcup \{W(C, n) : C \in F\}$. Since F is an $\mathfrak{I}$ -disjoint family, all W_n 's are $\mathfrak{I}$ -partitions of κ and form a descending sequence of partitions. If $\langle X_n : n < \omega \rangle$ is a path through $\langle W_n : n < \omega \rangle$, then $X_0 \in W(C, 0)$ for some $C \in F$ and since $X_n \subseteq X_0 \subseteq C$, $\langle X_n : n < \omega \rangle$ is a path through $\langle W(C, n) : n < \omega \rangle$, and hence its intersection is empty. $\square$

LEMMA 7. *Let κ, λ be uncountable cardinals. Let $\mathfrak{I}$ be a λ -complete nowhere precipitous ideal over κ . Then $\wp(\kappa)/\mathfrak{I}$ is $(\omega, \cdot, \lambda^+)$ -nowhere distributive.*

PROOF. By Lemma 6 there is a descending sequence $\langle W_n : N < \omega \rangle$ of $\mathfrak{I}$ -partitions of κ with no path through whose intersection is nonempty. We shall prove that

$$(\forall X \in \mathfrak{I}^+)(\exists n < \omega)(|\{Y \in W_n : X/\mathfrak{I} \wedge Y/\mathfrak{I} \neq \emptyset/\mathfrak{I}\}| \geq \lambda^+).$$

Then, by Lemma 4, $\wp(\kappa)/\mathfrak{I}$ is $(\omega, \cdot, \lambda^+)$ -nowhere distributive. By way of contradiction assume $(\exists X \in \mathfrak{I}^+)(\forall n < \omega)(|\{Y \in W_n : X/\mathfrak{I} \wedge Y/\mathfrak{I}\}| < \lambda^+)$. Since W_n is an $\mathfrak{I}$ -partition of κ , $Q_n = \{Y/\mathfrak{I} : Y \in W_n \text{ \& } X/\mathfrak{I} \wedge Y/\mathfrak{I} \neq \emptyset/\mathfrak{I}\}$ is a partition of $X/\mathfrak{I}$ of size $\leq \lambda$ (in the Boolean algebra $\wp(\kappa)/\mathfrak{I}$). For each $Y/\mathfrak{I} \in Q_n$

choose a representative and enumerate them, i.e. $Q_n = \{Y_\alpha^n / \mathfrak{S} : \alpha < \lambda_n\}$, where $\lambda_n = |Q_n| \leq \lambda$. Let $\hat{Q}_n = \{Y_\alpha^n : \alpha < \lambda_n\}$. Then $\hat{Q}_n$ is an $\mathfrak{S}$ -partition of X of size λ_n . Define $Z_0^n = Y_0^n$ and $Z_\alpha^n = Y_\alpha^n - \bigcup \{Y_\beta^n : \beta < \alpha\}$ for all $\alpha < \lambda_n$. Since $\mathfrak{S}$ is λ -complete and $\lambda_n \leq \lambda$, $\{Z_\alpha^n : \alpha < \lambda_n\}$ is a disjoint $\mathfrak{S}$ -partition of X refining $\hat{Q}_n$. Let $Z_n = X - \bigcup \{Z_\alpha^n : \alpha < \lambda_n\}$. Then every $Z_n \in \mathfrak{S}$ and so $X - \bigcup \{Z_n : n < \omega\} \neq \emptyset$, for $\mathfrak{S}$ is countable complete. Pick any $\rho \in X - \bigcup \{Z_n : n < \omega\}$. For every $n < \omega$, $\rho \in X$ and $\rho \notin Z_n$, hence there must be some $\alpha_n < \lambda_n$ so that $\rho \in Z_{\alpha_n}^n \subseteq Y_{\alpha_n}^n$. Since $\langle W_n : n < \omega \rangle$ is a descending sequence of $\mathfrak{S}$ -partitions of κ , $\langle Y_{\alpha_n}^n : n < \omega \rangle$ is a (descending) path through $\langle W_n : n < \omega \rangle$ which has a nonempty intersection (for it contains at least ρ), a contradiction. $\square$

LEMMA 8. *Let κ, λ be cardinals, $\kappa \geq 2$ and $\lambda \geq \omega$. Let B be a $(\lambda, \cdot, \kappa)$ -nowhere distributive Boolean algebra. Then*

- (1) *B is atomless and thus ω -closed,*
- (2) *if $D \subseteq B$ is dense in B , then $|D| \geq \kappa$,*
- (3) *$\text{Comp}(B)$ is also $(\lambda, \cdot, \kappa)$ -nowhere distributive.*

PROOF. Easy, left to the interested reader. $\square$

LEMMA 9. *Let κ, λ be uncountable cardinals. Let $\mathfrak{S}$ be a λ -complete nowhere precipitous ideal over κ so that $\wp(\kappa)/\mathfrak{S}$ has a dense set of size λ^+ . Then $\text{Comp}(\wp(\kappa)/\mathfrak{S})$ and $\text{Col}(\omega, \lambda^+)$ are isomorphic.*

PROOF. Since $\mathfrak{S}$ is λ -complete and nowhere precipitous, by Lemma 6, $\wp(\kappa)/\mathfrak{S}$ is $(\omega, \cdot, \lambda^+)$ -nowhere distributive. By Lemma 8, $\text{Comp}(\wp(\kappa)/\mathfrak{S})$ is also $(\omega, \cdot, \lambda^+)$ -nowhere distributive and $d(\text{Comp}(\wp(\kappa)/\mathfrak{S})) \geq \lambda^+$. Since $\wp(\kappa)/\mathfrak{S}$ has a dense set of size λ^+ , so does $\text{Comp}(\wp(\kappa)/\mathfrak{S})$. Hence $d(\text{Comp}(\wp(\kappa)/\mathfrak{S})) = \lambda^+ = (\lambda^+)^{<\omega}$. By Lemma 8, $\wp(\kappa)/\mathfrak{S}$ is ω -closed. Thus all requirements of Lemma 5 are satisfied and so $\text{Comp}(\wp(\kappa)/\mathfrak{S})$ and $\text{Col}(\omega, \lambda^+)$ are isomorphic. $\square$

These were preliminaries needed to deal with ideals over regular cardinals. Let us now turn our attention to singular cardinals. Unfortunately, we can only describe the structure of $\text{Comp}(\wp(\kappa)/\mathfrak{S}_\kappa)$.

Let $\omega \leq \lambda = \text{cf}(\kappa) < \kappa$. (Recall that λ is regular, $\wp(\lambda)/\mathfrak{S}_\lambda$ can be regularly imbedded into $\wp(\kappa)/\mathfrak{S}_\kappa$, and $\mathfrak{S}_\kappa$ is λ -complete.)

As usual, for $X, Y \in [\lambda]^\lambda$ define $X \subseteq^* Y$ iff $(X - Y) \in \mathfrak{S}_\lambda$, i.e. iff $|X - Y| < \lambda$.

LEMMA 10. *Let $\lambda \geq \omega$ be regular. For every $X \in [\lambda]^\lambda$ define by induction $h_X(\alpha) = \min(\lambda - \bigcup \{h_X(\beta) : \beta < \alpha\})$ for all $\alpha \in X$. Then*

- (1) *h_X is strictly increasing,*
- (2) *$(\forall \alpha \in X)(h_X(\alpha) \leq \alpha)$,*
- (3) *if $Y \in [\lambda]^\lambda$ and $Y \subseteq^* X$, then*

$$(\exists \beta \in X \cap Y)(\forall \alpha \in X \cap Y)(\alpha \geq \beta \rightarrow h_X(\alpha) \geq h_Y(\alpha)),$$

- (4) *if $s : X \rightarrow \lambda$ is strictly increasing, then $(\forall \alpha \in X)(h_X(\alpha) \leq s(\alpha))$.*

PROOF. Standard, left to the reader. (Note that h_X is the inverse of the enumeration function for X .) $\square$

LEMMA 11. *Let $\omega \leq \lambda = \text{cf}(\kappa) < \kappa$. Assume that there is a sequence $\langle P_\alpha : \alpha \in \mu \rangle$ of $\mathfrak{S}_\lambda$ -partitions of λ so that $(\forall X \in [\lambda]^\lambda)(\exists \alpha \in \mu)(\exists Y \in P_\alpha)(Y \subseteq^* X)$. Then $\wp(\kappa)/\mathfrak{S}_\kappa$ is $(\mu, \cdot, \kappa^+)$ -nowhere distributive.*

PROOF. Let $\kappa = \bigcup\{Q_\alpha : \alpha \in \lambda\}$ where $\langle Q_\alpha : \alpha \in \lambda \rangle$ is a disjoint family of subsets of κ so that $\langle |Q_\alpha| : \alpha \in \lambda \rangle$ is a strictly increasing sequence of regular cardinals and $\kappa = \sum\{|Q_\alpha| : \alpha \in \lambda\}$. Let $\kappa_\alpha = |Q_\alpha|$ for every $\alpha \in \lambda$. For any $X \in [\lambda]^\lambda$ define $\hat{X} = \bigcup\{Q_\alpha : \alpha \in X\}$. Let $h_X : X \rightarrow \lambda$ be as in Lemma 10. Define

$$S_X = \{D \in [\kappa]^\kappa : (\forall \alpha \in X)(|D \cap Q_\alpha| \leq |Q_{h_X(\alpha)}|) \& (\forall \alpha \in (\lambda - X))(D \cap Q_\alpha = \emptyset)\}.$$

Claim 1. For any $X \in [\lambda]^\lambda$, S_X is dense in $([\hat{X}]^\kappa, \subseteq)$.

If $D \in S_X$, then $D = \bigcup\{D \cap Q_\alpha : \alpha \in X\} \subseteq \bigcup\{Q_\alpha : \alpha \in X\} = \hat{X}$, so $D \in [\hat{X}]^\kappa$. Let $Y \in [\hat{X}]^\kappa$. Then

$$(\forall \gamma \in \lambda)(\forall \beta \in X)(\exists \alpha \in X)(\alpha > \beta \& |Y \cap Q_\alpha| \geq |Q_\gamma|).$$

If not, then $(\exists \gamma \in \lambda)(\exists \beta \in X)(\forall \alpha \in X)(\alpha > \beta \rightarrow |Y \cap Q_\alpha| < |Q_\gamma|)$ and so $|Y| < |Q_\beta| + |G_\gamma| < \kappa$, a contradiction. Thus we can define by induction a strictly increasing function $f : \lambda \rightarrow X$ so that $|Y \cap Q_{f(\gamma)}| \geq |Q_\gamma|$ for all $\gamma \in \lambda$ and $\gamma \leq h_X(f(\gamma)) \leq f(\gamma)$. For every $\alpha \in X$ define D_α by

(i) if $\alpha = f(\gamma)$ for some $\gamma \in \lambda$, then let D_α be a subset of $(Y \cap Q_{f(\gamma)})$ of size $\geq |Q_\gamma|$ but $\leq |Q_{h_X(f(\gamma))}|$. Notice that this is possible as $\gamma \leq h_X(f(\gamma))$ and so $|Q_\gamma| \leq |Q_{h_X(f(\gamma))}|$,

(ii) otherwise $D_\alpha = \emptyset$.

Let $D = \bigcup\{D_\alpha : \alpha \in X\}$. Then $D \subseteq Y$, and $|D| \geq \sum\{|Q_\gamma| : \gamma \in \text{rng}(f)\} = \kappa$ as $\text{rng}(f) = \lambda$, and also $|D \cap Q_\alpha| \leq |Q_{h_X(\alpha)}|$ for all $\alpha \in X$. Hence $D \in S_X$ and so the claim is proven.

By Claim 1, for any $X \in [\lambda]^\lambda$ there is an almost disjoint partition of $\hat{X}$ consisting entirely of elements of S_X . Choose one and denote it R_X . Define

$$W_\alpha = \bigcup\{R_X : X \in P_\alpha\}$$

for every $\alpha \in \mu$.

Claim 2. Each W_α is an almost disjoint partition of κ .

Clearly, for each $Z \in W_\alpha$, $|Z| = \kappa$. Now let $Z_1, Z_2 \in W_\alpha$. If $Z_1, Z_2 \in R_X$ for some $X \in P_\alpha$, then $|Z_1 \cap Z_2| < \kappa$. So assume that $Z_1 \in R_{X_1}, Z_2 \in R_{X_2}$, and $X_1 \neq X_2 \in P_\alpha$. Assume by way of contradiction that $|Z_1 \cap Z_2| = \kappa$. If $\beta \in Z_1 \cap Z_2$, then $\beta \in \hat{X}_1 \cap \hat{X}_2$, so $\beta \in Q_\gamma$ for some $\gamma \in X_1 \cap X_2$. Thus there is a function $f : (Z_1 \cap Z_2) \rightarrow (X_1 \cap X_2)$ so that $\beta \in Q_{f(\beta)}$. Since $|X_1 \cap X_2| < \lambda$, and hence $|\text{rng}(f)| < \lambda$, there are $\gamma \in X_1 \cap X_2$ and $Z_3 \in [Z_1 \cap Z_2]^\kappa$ such that $f(\beta) = \gamma$ for all $\beta \in Z_3$. Then $Z_3 \subseteq Q_\gamma$, so $|Q_\gamma| = \kappa$, a contradiction. Hence $|Z_1 \cap Z_2| < \kappa$. The claim is proven.

Claim 3. $\{W_\alpha : \alpha \in \mu\}$ witnesses that $\wp(\kappa)/\mathfrak{S}_\kappa$ is $(\mu, \cdot, \kappa^+)$ -nowhere distributive.

Let $C \in [\kappa]^\kappa$. Then $(\forall \gamma \in \lambda)(\forall \beta \in \lambda)(\exists \alpha \in \lambda)(\alpha > \beta \& |C \cap Q_\alpha| > |Q_\gamma|)$. If not, then $(\exists \gamma \in \lambda)(\exists \beta \in \lambda)(\forall \alpha \in \lambda)(\alpha > \beta \rightarrow |C \cap Q_\alpha| \leq |Q_\gamma|)$. Then $|C| \leq |Q_\beta| + |Q_\gamma| < \kappa$, a contradiction. Thus there is a strictly increasing sequence $\langle \alpha_\gamma : \gamma \in \lambda \rangle \subseteq \lambda$ so that $|C \cap Q_{\alpha_\gamma}| > |Q_\gamma|$. Let $X = \{\alpha_\gamma : \gamma \in \lambda\}$. So $X \in [\lambda]^\lambda$. Define $s(\alpha_\gamma) = \gamma$ for all $\gamma \in \lambda$. Thus $s : X \rightarrow \lambda$ is strictly increasing and

therefore $h_X(\alpha) \leq s(\alpha)$ for all $\alpha \in X$. By the assumption of this lemma, $(\exists \delta \in \mu)(\exists Y \in P_\delta)(Y \subseteq^* X)$. Then there is $\bar{\alpha} \in X \cap Y$ so that $h_Y(\alpha) \leq h_X(\alpha)$ for all $\alpha \in (X \cap Y) - \bar{\alpha}$. Recall that R_Y is an almost disjoint partition of $\hat{Y}$ consisting of elements of S_Y , and also $R_Y \subseteq W_\delta$. If $Z \in R_Y$, then $Z \in [\kappa]^\kappa$, $(\forall \alpha \in Y)(|Z \cap Q_\alpha| \leq |Q_{h_Y(\alpha)}|)$ and $(\forall \alpha \in \lambda - Y)(Z \cap Q_\alpha = \emptyset)$.

(*) There are at least κ^+ elements of R_Y which have an intersection of size κ with C .

If not, let $\{Z_\rho: \rho \in \kappa\}$ be a list of elements of R_Y which contains all $Z \in R_Y$ so that $|Z \cap C| = \kappa$ (if need be, with repetitions). Let $D = C \cap \hat{X} \cap \hat{Y}$. Then

(**) $(\forall \beta \in \lambda)(\exists \gamma \in \lambda) \left(\gamma > \beta \ \& \ \left| \left(D - \bigcup \{Z_\rho: \rho \in \kappa_\beta\} \right) \cap Q_{\alpha_\gamma} \right| > |Q_\gamma| \right)$.

Let $\beta \in \lambda$. Since $Y \subseteq^* X$, $|X \cap Y| = \lambda$. There is $\gamma \in \lambda$ so that $\gamma > \beta$ and $\alpha_\gamma \in (X \cap Y) - \bar{\alpha}$. If $\rho < \kappa_\beta$, then $|Z_\rho \cap Q_{\alpha_\gamma}| \leq |Q_{h_Y(\alpha_\gamma)}| \leq |Q_{h_X(\alpha_\gamma)}| \leq |Q_{s(\alpha_\gamma)}| = |Q_\gamma|$. Thus

$$\left| \bigcup \{Z_\rho: \rho < \kappa_\beta\} \cap Q_{\alpha_\gamma} \right| \leq \kappa_\beta \times |Q_\gamma| = |Q_\gamma|$$

as $\gamma > \beta$ and $|Q_\gamma| = \kappa_\gamma$.

Since $\alpha_\gamma \in X \cap Y$, $Q_{\alpha_\gamma} \subseteq \hat{X} \cap \hat{Y}$. $|D \cap Q_{\alpha_\gamma}| = |(C \cap \hat{X} \cap \hat{Y}) \cap Q_{\alpha_\gamma}| = |C \cap Q_{\alpha_\gamma}| > |Q_\gamma|$. Thus

$$\begin{aligned} & \left| \left(D - \bigcup \{Z_\rho: \rho \in \kappa_\beta\} \right) \cap Q_{\alpha_\gamma} \right| \\ &= \left| (D \cap Q_{\alpha_\gamma}) - \left(\bigcup \{Z_\rho: \rho < \kappa_\beta\} \cap Q_{\alpha_\gamma} \right) \right| > |Q_\gamma|. \end{aligned}$$

Therefore (**) holds.

By (**) there is a strictly increasing sequence $\langle \gamma_\beta: \beta < \lambda \rangle \subseteq \lambda$ so that $|D - \bigcup \{Z_\rho: \rho \in \kappa_\beta\} \cap Q_{\alpha_{\gamma_\beta}}| > |Q_{\gamma_\beta}|$. For each $\beta \in \lambda$ let A_β be a subset of $(D - \bigcup \{Z_\rho: \rho \in \kappa_\beta\}) \cap Q_{\alpha_{\gamma_\beta}}$ of size $|Q_{\gamma_\beta}|$. Let $A = \bigcup \{A_\beta: \beta \in \lambda\}$. Then $A \subseteq D$, so $A \subseteq \hat{Y}$ and $A \subseteq C$. $|A| = \sum \{|A_\beta|: \beta \in \lambda\} = \sum \{|Q_{\gamma_\beta}|: \beta \in \lambda\} = \sum \{\kappa_{\gamma_\beta}: \beta \in \lambda\} = \kappa$. So $A \in |\hat{Y}|^\kappa$. Since R_Y is an almost disjoint partition of $\hat{Y}$, for some $Z \in R_Y$, $|Z \cap A| = \kappa$. Then also $|Z \cap C| = \kappa$. Therefore $Z = Z_{\rho_0}$ for some $\rho_0 \in \kappa$. Let $\beta \in \lambda$ so that $\rho_0 < \kappa_\beta$ and $Z \cap A_\beta \neq \emptyset$. Let $x \in Z \cap A_\beta$. Then $x \notin \bigcup \{Z_\rho: \rho < \kappa_\beta\}$, so $x \notin Z_{\rho_0} = Z$, a contradiction. Hence (*) holds.

Thus Claim 3 is proven.

This completes the proof of the lemma. $\square$

LEMMA 12. *Let $\lambda \geq \omega_1$ be regular. Let $B = \text{Col}(\omega, \lambda^+)$. There is a descending sequence $\langle P_n: n < \omega \rangle$ of partitions of 1_B such that $(\forall b \in B^+)(\exists n < \omega)(\exists c \in P_n)(c < b)$ and so that $\bigcup \{P_n: n < \omega\}$ is dense in B .*

PROOF [SKETCH]. Let $C = {}^{<\omega}(\lambda^+)$. For $f, g \in C$ let $f \leq g$ iff $g \subseteq f$. Then $B = \text{Comp}((C, \leq))$. For every $n < \omega$, let $P_n = {}^n(\lambda^+)$. Now it is easy to check that $\langle P_n: n < \omega \rangle$ is a required sequence. $\square$

LEMMA 13. *Let $2^\omega = \omega_1$. There is a sequence $\langle P_\alpha : \alpha \in \omega_1 \rangle$ of $\mathfrak{I}_\omega$ -partitions of ω such that $(\forall X \in [\omega]^\omega)(\exists \alpha \in \omega_1)(\exists Y \in P_\alpha)(Y \subseteq X)$.*

PROOF. Easy, left to the reader (or follows from Base Matrix Theorem in [BPS] (see [BVo])).

LEMMA 14. *Let $\omega = cf(\kappa) < \kappa$. Then $\wp(\kappa)/\mathfrak{I}_\kappa$ is ω_1 -closed.*

PROOF. Let $\langle \kappa_n : n < \omega \rangle$ be an increasing sequence of regular cardinals cofinal in κ . $\wp(\kappa)/\mathfrak{I}_\kappa$ is atomless, for if $X \notin \mathfrak{I}_\kappa$, then $|X| = \kappa$ and so there is $Y \subseteq X$ so that $|Y| = \kappa$ and $|X - Y| = \kappa$, hence $\emptyset/\mathfrak{I}_\kappa < Y/\mathfrak{I}_\kappa < X/\mathfrak{I}_\kappa$ and so $X/\mathfrak{I}_\kappa$ is not an atom. Thus $\wp(\kappa)/\mathfrak{I}_\kappa$ is ω -closed. Let $\langle X_n/\mathfrak{I}_\kappa : n < \omega \rangle$ be a strictly decreasing sequence of elements of $\wp(\kappa)/\mathfrak{I}_\kappa$. Let $Y_n = \bigcap \{X_m : m \leq n\}$ for all $n < \omega$. Since $X_n = Y_n \cup \bigcup \{X_n - X_m : m < n\}$ and $\bigcup \{X_n - X_m : m < n\} \in \mathfrak{I}_\kappa$, $Y_n/\mathfrak{I}_\kappa = X_n/\mathfrak{I}_\kappa$ for every $n < \omega$. Thus $\langle Y_n/\mathfrak{I}_\kappa : n < \omega \rangle$ is a strictly decreasing sequence of elements of $\wp(\kappa)/\mathfrak{I}_\kappa$, and therefore $|Y_{n+1} - Y_n| = \kappa$ for all $n < \omega$. For each $n < \omega$ choose some $A_n \subset (Y_n - Y_{n+1})$ of size κ_n . Let $A = \bigcup \{A_n : n < \omega\}$. Then $|A| = \sum \{\kappa_n : n < \omega\} = \kappa$, $A \subseteq Y_0$, and for every $n \geq 1$, $A - Y_n = \bigcup \{A_i : i < n\}$, hence $A \subseteq^* Y_n$ and so $A \subseteq^* X_n$ for all $n < \omega$. Thus $\emptyset/\mathfrak{I}_\kappa < A/\mathfrak{I}_\kappa \leq X_n/\mathfrak{I}_\kappa$ for all $n < \omega$. $\square$

4. Main results.

PROOF OF THEOREM 1. By Lemma 7, $\wp(\kappa)/\mathfrak{I}$ is $(\omega, \cdot, \kappa^+)$ -nowhere distributive and so by Lemma 8 $d(\wp(\kappa)/\mathfrak{I}) \geq \kappa^+$. Since $|\wp(\kappa)/\mathfrak{I}| \leq |\wp(\kappa)| = 2^\kappa = \kappa^+$, $d(\wp(\kappa)/\mathfrak{I}) = \kappa^+$ and so $\wp(\kappa)/\mathfrak{I}$ has a dense set of size κ^+ . Thus, by Lemma 9, $\text{Comp}(\wp(\kappa)/\mathfrak{I})$ and $\text{Col}(\omega, \kappa^+)$ are isomorphic. $\square$

PROOF OF COROLLARY 2. It is known that under $V = L$, there are no measurable cardinals, and hence no countably complete precipitous ideals (see e.g. [J]) and that the G.C.H. holds. Therefore for any κ -complete ideal $\mathfrak{I}$ over κ , all requirements of Theorem 1 are satisfied, and so $\text{Comp}(\wp(\kappa)/\mathfrak{I})$ and $\text{Col}(\omega, \kappa^+)$ are isomorphic. $\square$

NOTE. Why cannot Theorem 1 be applied to singular cardinals? For if $\mathfrak{I}$ is a κ -complete ideal over a singular cardinal κ , then $\kappa \in \mathfrak{I}$ and it is a contradiction. Is it necessary that $\mathfrak{I}$ be κ -complete? As long as we can find a dense set of size λ^+ in $\wp(\kappa)/\mathfrak{I}$, we need just λ -completeness (see Lemma 9). Thus we need κ -completeness only to get “close” to the “natural” estimate of the size of $\wp(\kappa)/\mathfrak{I}$.

PROOF OF THEOREM 3. Let κ be a singular cardinal. Assume $2^\kappa = \kappa^+$ and $2^{cf(\kappa)} = cf(\kappa)^+$.

(1) Assume that $cf(\kappa) = \omega$.

By Lemmas 13 and 11, $\wp(\kappa)/\mathfrak{I}_\kappa$ is $(\omega_1, \cdot, \kappa^+)$ -nowhere distributive. By Lemma 8, $\text{Comp}(\wp(\kappa)/\mathfrak{I}_\kappa)$ is $(\omega_1, \cdot, \kappa^+)$ -nowhere distributive. By Lemma 14, $\wp(\kappa)/\mathfrak{I}_\kappa$ is ω_1 -closed. By Lemma 8, $d(\text{Comp}(\wp(\kappa)/\mathfrak{I}_\kappa)) \geq \kappa^+$. Since $|\wp(\kappa)/\mathfrak{I}_\kappa| \leq 2^\kappa = \kappa^+$, $d(\text{Comp}(\wp(\kappa)/\mathfrak{I}_\kappa)) = \kappa^+$. Since $2^\kappa = \kappa^+$ and $cf(\kappa) = \omega$, $(\kappa^+)^{<\omega_1} = \kappa^+$. Therefore, by Lemma 5, $\text{Comp}(\wp(\kappa)/\mathfrak{I}_\kappa)$ and $\text{Col}(\omega_1, \kappa^+)$ are isomorphic.

(2) Assume that $\lambda = cf(\kappa) > \omega$.

By Theorem 1, $\text{Comp}(\wp(\lambda)/\mathfrak{I}_\lambda)$ is isomorphic to $\text{Col}(\omega, \lambda^+)$, for $\mathfrak{I}_\lambda$ is nowhere precipitous (see [J] or [JP]), λ -complete and $2^\lambda = \lambda^+$. Thus, by Lemma 12,

there is a descending sequence $\langle P_n : n < \omega \rangle$ of $\mathfrak{S}_\lambda$ -partitions of λ so that $(\forall X \in [\lambda]^\lambda)(\exists n < \omega)(\exists Y \in P_n)(Y \subseteq^* X)$. Hence, by Lemma 11, $\wp(\kappa)/\mathfrak{S}_\kappa$ is $(\omega, \cdot, \kappa^+)$ -nowhere distributive. By Lemma 8, $\text{Comp}(\wp(\kappa)/\mathfrak{S}_\kappa)$ is $(\omega, \cdot, \kappa^+)$ -nowhere distributive, $d(\text{Comp}(\wp(\kappa)/\mathfrak{S}_\kappa)) \geq \kappa^+$, and $\wp(\kappa)/\mathfrak{S}_\kappa$ is ω -closed. Since $|\wp(\kappa)/\mathfrak{S}_\kappa| \leq 2^\kappa = \kappa^+$, $d(\text{Comp}(\wp(\kappa)/\mathfrak{S}_\kappa)) = \kappa^+$. Clearly $(\kappa^+)^{<\omega} = \kappa^+$ and so, by Lemma 5, $\text{Comp}(\wp(\kappa)/\mathfrak{S}_\kappa)$ and $\text{Col}(\omega, \kappa^+)$ are isomorphic. $\square$

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