ON n -DIMENSIONAL LORENTZ MANIFOLDS ADMITTING AN ISOMETRY GROUP OF DIMENSION $n(n-1)/2+1$

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ABSTRACT. We classify connected n -dimensional Lorentz manifolds admitting an isometry group of dimension $n(n-1)/2+1$ with compact isotropy subgroup ($n \geq 5$).

1. A connected n -dimensional Riemannian manifold admitting a connected closed isometry group of dimension $n(n-1)/2+1$ ($n \geq 4$) was completely determined by Yano [8], Ishihara [1], and Obata [5] (cf. Kobayashi [2]). We consider the classification problem of Lorentz manifolds. Each of the following examples is a connected n -dimensional Lorentz manifold M admitting a connected isometry group G of dimension $n(n-1)/2+1$.

EXAMPLE. (i) $M = \mathbf{R} \times N$ with metric $-dt^2 + ds_N^2$, and $G = \mathbf{R} \times I^0(N)$.

(ii) $M = S^1 \times N$ with metric $-d\theta^2 + ds_N^2$, and $G = S^1 \times I^0(N)$.

(iii) $M = \mathbf{R} \times \mathbf{P}$ with metric $-dt^2 + ds_{\mathbf{P}}^2$, and $G = \mathbf{R} \times I^0(\mathbf{P})$.

(iv) $M = S^1 \times \mathbf{P}$ with metric $-d\theta^2 + ds_{\mathbf{P}}^2$, and $G = S^1 \times I^0(\mathbf{P})$.

(v) $M = U_n^+ = \{(u_1, \dots, u_n); u_n > 0\}$ with metric

$$ds_+^2 = (du_1^2 + \dots + du_{n-1}^2 - du_n^2)/(cu_n)^2 \quad (c \neq 0),$$

and $G = I^0(U_n^+)$ (see Nomizu [4]).

(vi) $M = U_n^- = \{(u_1, \dots, u_n); u_n > 0\}$ with metric

$$ds_-^2 = (-du_1^2 + du_2^2 + \dots + du_n^2)/(cu_n)^2 \quad (c \neq 0),$$

and $G = I^0(U_n^-)$ (see Matsuda [3]).

Here N is a simply connected $(n-1)$ -dimensional Riemannian manifold with metric ds_N^2 of constant curvature and $\mathbf{P}$ is an $(n-1)$ -dimensional real projective space with standard metric $ds_{\mathbf{P}}^2$. A real line and a circle of a certain radius are denoted by $\mathbf{R}$ and S^1 respectively. $I^0(\cdot)$ denotes the identity component of the full-isometry group of $(\cdot)$.

The purpose of this note is to prove the following theorem.

THEOREM. *Let M be a connected n -dimensional Lorentz manifold admitting a connected isometry group G of dimension $n(n-1)/2+1$ ($n \geq 5$) whose isotropy subgroup at every point is compact. Then M must be one of the spaces (i)–(v).*

REMARK 1. The isotropy subgroup of G in the above example is compact except (vi).

REMARK 2. The spaces of (v) and (vi) are not geodesically complete.

Received by the editors May 1, 1985.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 53C50.

Key words and phrases. Lorentz manifold, isometry group, semi-Riemannian submersion.

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0002-9939/87 \$1.00 + \$.25 per page

2. Let $(M, \langle \cdot, \cdot \rangle)$ be a connected n -dimensional Lorentz manifold with signature $(-, +, \dots, +)$. Let G be a connected isometry group of $(M, \langle \cdot, \cdot \rangle)$ and H_p the isotropy subgroup of G at a point $p \in M$. Then the linear isotropy group $\tilde{H}_p = \{dh_p; h \in H_p\}$ acting on $T_p M$ is a closed subgroup of $O(1, n-1)$.

LEMMA 1. *Every compact subgroup of $O(1, n-1)$ is conjugate to a subgroup of $O(1) \times O(n-1)$ (cf. Wolf [7]). Especially if K is a compact subgroup of $O(1, n-1)$ whose dimension is $(n-1)(n-2)/2$, then K leaves invariant one and only one 1-dimensional subspace in an n -dimensional vector space (cf. Obata [5]).*

We can see that for $n(n+1)/2 \geq r > n(n-1)/2 + 1$ the full-isometry group of M contains no subgroup of r whose isotropy subgroup is compact. In fact, suppose that for $n(n+1)/2 \geq r > n(n-1)/2 + 1$ there exists an r -dimensional subgroup G whose isotropy subgroup H_p is compact for some $p \in M$. Then $\dim H_p \leq (n-1)(n-2)/2$ from Lemma 1. On the other hand, we have

$$\dim H_p \geq \dim G - \dim M > n(n-1)/2 + 1 - n = (n-1)(n-2)/2$$

which is a contradiction to the fact that $\dim H_p \leq (n-1)(n-2)/2$.

We can also derive the following proposition from Lemma 1.

PROPOSITION. *If M admits a connected isometry group G of dimension $n(n-1)/2 + 1$ whose isotropy subgroup H_p is compact for every $p \in M$, then G is transitive on M .*

PROOF. Assume that G is not transitive on M . Then the orbit of G through p is of dimension less than n . Hence,

$$\dim H_p \geq \dim G - (n-1) = (n-1)(n-2)/2 + 1$$

which is impossible from Lemma 1.

Hereafter, let G be a connected isometry group of dimension $n(n-1)/2 + 1$ whose isotropy subgroup H_p is compact for every $p \in M$. From the proposition, $\dim H_p = (n-1)(n-2)/2$ and we may assume that $M = G/H_p$.

Let $\mathfrak{g}$ and $\mathfrak{h}$ be the Lie algebras of G and $H := H_o$ respectively. But the use of an $\text{Ad}(H)$ -invariant positive definite inner product on $\mathfrak{g}$ which exists from the compactness of H , we have a decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ (direct sum) of $\mathfrak{g}$ where $\text{Ad}(H)\mathfrak{m} \not\subseteq \mathfrak{m}$. Let $\pi: G \rightarrow G/H$ be the natural projection. We identify the tangent space $T_o M$ at $o := \pi(H)$ and $\mathfrak{m}$ by $d\pi$. Then the linear isotropy group $\tilde{H}$ acting on $T_o M$ corresponds under $d\pi$ to $\text{Ad}(H)$ on $\mathfrak{m}$. The Lorentz inner product $T_o M$ induces the Lorentz inner product $\langle \cdot, \cdot \rangle_{\mathfrak{m}}$ on $\mathfrak{m}$ so that $d\pi: \mathfrak{m} \rightarrow T_o M$ is a linear isometry. We note that the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{m}}$ on $\mathfrak{m}$ is $\text{Ad}(H)$ -invariant.

We define the Lorentz inner product B on $\mathfrak{g}$ such that

$$B(\mathfrak{h}, \mathfrak{m}) = 0, \quad B \upharpoonright_{\mathfrak{m}} = \langle \cdot, \cdot \rangle_{\mathfrak{m}},$$

and $B \upharpoonright_{\mathfrak{h}}$ is positive definite. We extend B to the G -left invariant Lorentz metric on G which is denoted by the same letter B . Then (G, B) is a Lorentz manifold and $\pi: (G, B) \rightarrow (M, \langle \cdot, \cdot \rangle)$ is a semi-Riemannian submersion (see O'Neill [6, p. 212]).

Since $\text{Ad}(H)$ acts on $\mathfrak{m}$ as a compact linear isometry group and $\dim \text{Ad}(H) = (n-1)(n-2)/2$, it follows from Lemma 1 that there exists one and only one 1-dimensional subspace $\mathfrak{m}_1$ of $\mathfrak{m}$ which is invariant by $\text{Ad}(H)$. Furthermore, we have

LEMMA 2. $\mathfrak{m}_1$ is timelike.

Putting $\mathfrak{m}_2 = \{X \in \mathfrak{m}; B(X, \mathfrak{m}_1) = 0\}$, we have a direct sum decomposition $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$ of $\mathfrak{m}$ where $[\mathfrak{h}, \mathfrak{m}_1] = (0)$, $[\mathfrak{h}, \mathfrak{m}_2] = \mathfrak{m}_2$, and the adjoint representation of $\mathfrak{h}$ in $\mathfrak{m}_2$ is irreducible. Thus we have a decomposition $\mathfrak{g} = \mathfrak{m}_1 + \mathfrak{m}_2 + \mathfrak{h}$ of $\mathfrak{g}$ by $\text{ad}(\mathfrak{h})$, where $[\mathfrak{h}, \mathfrak{m}_1] = (0)$ and $[\mathfrak{h}, \mathfrak{m}_2] = \mathfrak{m}_2$ (see Obata [5]).

LEMMA 3 (OBATA [5]). We have the following possibilities provided $n \geq 5$:

- (i) $[\mathfrak{h}, \mathfrak{m}_1] = (0)$, $[\mathfrak{m}_1, \mathfrak{m}_2] = (0)$, $[\mathfrak{m}_2, \mathfrak{m}_2] = (0)$;
- (ii) $[\mathfrak{h}, \mathfrak{m}_1] = (0)$, $[\mathfrak{m}_1, \mathfrak{m}_2] = (0)$, $[\mathfrak{m}_2, \mathfrak{m}_2] = \mathfrak{h}$;
- (iii) $[\mathfrak{h}, \mathfrak{m}_1] = (0)$, $[\mathfrak{m}_1, \mathfrak{m}_2] = \mathfrak{m}_2$, $[\mathfrak{m}_2, \mathfrak{m}_2] = (0)$, and $[X, Y] = L(X)Y$ for any $X \in \mathfrak{m}_1$ and $Y \in \mathfrak{m}_2$ where L is the linear function on $\mathfrak{m}_1$ such that $L(X) \neq 0$ for any nonzero $X \in \mathfrak{m}_1$.

REMARK 3. For a unit vector $E \in \mathfrak{m}_1$ (i.e. $B(E, E) = -1$) we put $c := L(E)$. We may assume $c > 0$.

We set $\mathfrak{g}' = \mathfrak{h} + \mathfrak{m}_2$ and $B' = B|_{\mathfrak{g}'}$. Since B' is positive definite and $\text{ad}(\mathfrak{h})|_{\mathfrak{m}_2} = o(n-1)$, there is a basis $\{X_i, X_{jk}\}$ ($1 \leq i \leq n-1$, $1 \leq j < k \leq n-1$) of $\mathfrak{g}'$ such that

- (1) $\{X_i\}$ (resp. $\{X_{jk}\}$) is a basis of $\mathfrak{m}_2$ (resp. $\mathfrak{h}$),
- (2) $B'(X_k, X_j) = \delta_{ij}$,
- (3) $[X_{ij}, X_k] = \delta_{ik}X_j - \delta_{jk}X_i$ ($1 \leq i < j \leq n-1$, $1 \leq k \leq n-1$).

Then

LEMMA 4 (OBATA [5]). In case (ii) in Lemma 3, there exists a nonzero constant α such that $[X_i, X_j] = \alpha X_{ij}$ ($1 \leq i < j \leq n-1$).

3. From Lemma 3, $\mathfrak{m}_1$ (resp. $\mathfrak{m}_2$) induces an integrable G -invariant 1- (resp. $n-1$ -) dimensional distribution T_1 (resp. T_2) on M such that at each point p of M , $T_p M = T_1(p) + T_2(p)$, $\langle T_1(p), T_2(p) \rangle = 0$, and $T_1(p)$ (resp. $T_2(p)$) is timelike (resp. spacelike).

Now, we assume that M is simply connected.

LEMMA 5. When M is simply connected, $\xi := d\pi(E)$ is well defined on M .

PROOF. We will show that for each $p \in M$, $\xi(p) = d\pi(E(g)) = d\tau_g d\pi(E(e))$ is independent of the choice of $g \in G$ such that $go = p$, where e is the identity of G , $o = \pi(H)$, and τ_g is the map: $x \rightarrow gx$ on M .

Let $g_1 o = g_2 o = p$ ($g_1, g_2 \in G$). G being connected, there exist curves $\tilde{g}_i: [0, 1] \rightarrow G$ such that $\tilde{g}_i(0) = e$ and $\tilde{g}_i(1) = g_i$ ($i = 1, 2$). Set $c_i(t) := \tilde{g}_i(t)o$ ($i = 1, 2$). M being simply connected, M is time orientable. So there exists a unit timelike vector field X on M . Then we can see that

$$\langle X(c_i(t)), d\tau_{\tilde{g}_i(t)} \xi(o) \rangle \neq 0 \quad \text{for any } t \in [0, 1].$$

The map: $t \rightarrow \langle X(c_i(t)), d\tau_{\tilde{g}_i(t)} \xi(o) \rangle$ being continuous, if $\langle X(o), \xi(o) \rangle < 0$ (resp. > 0), then $\langle X(p), d\tau_{g_i} \xi(o) \rangle < 0$ (resp. > 0). Thus $d\tau_{g_1} \xi(o)$ and $d\tau_{g_2} \xi(o)$ belong to the same connected component of the time cone in $T_p M$. Furthermore, $d\tau_{g_1} \xi(o)$ and $d\tau_{g_2} \xi(o)$ belong to $T_1(p)$. Therefore $d\tau_{g_1} \xi(o) = d\tau_{g_2} \xi(o)$.

We define the nonzero 1-form ω on M by $\omega(X) := \langle X, \xi \rangle$. Then we can see ω is closed. Since M is simply connected there exists a C^∞ function $f: M \rightarrow \mathbb{R}$ such that $df = \omega$. For each $a \in f(M)$, $f^{-1}(a)$ is a closed spacelike hypersurface

of M and each connected component of $f^{-1}(a)$ is a leaf of T_2 . Since $\nabla_\xi \xi = 0$ and $\xi = d\pi(E)$, each integral curve of ξ which is a leaf of T_1 is a complete geodesic. For a point $p \in M$, let $\gamma_p(t)$ be the integral curve of ξ such that $\gamma_p(0) = p$. Then we have easily that $f(\gamma_p(t)) = -t + a$ for $p \in f^{-1}(a)$. Therefore $f(M) = \mathbf{R}$. Let N be $f^{-1}(0)$ and N_0 be a connected component of N . We define the map $F: \mathbf{R} \times N_0 \rightarrow M$ by $F(t, x) := \gamma_x(t) = \text{Exp}(t\xi(x))$. Then we have

LEMMA 6. F is the onto diffeomorphism.

PROOF. Assume that $F(t, x) = F(t', x')$. We have

$$t = -f(\gamma_x(t)) = -f(F(t, x)) = -f(F(t', x')) = t'.$$

Since geodesics γ_x and $\gamma_{x'}$ are leaves of T_1 through $F(t, x) = F(t', x')$ and $t = t'$, we have $x = x'$. Thus F is one-to-one. It is evident that F is differentiable. Set $M_0 := F(\mathbf{R} \times N_0)$. Then M_0 is open in M . It remains to be shown that M_0 is closed in M . Suppose that $F(t_k, x_k) = p_k$ is a sequence approaching some point q in M . Let $\tilde{f}: \mathbf{R} \rightarrow \mathbf{R}$ be the function defined by $\tilde{f}(t) := f(F(t, x))$ for some $x \in N_0$. Then $\tilde{f}$ is independent of the choice $x \in N_0$, for $\tilde{f}(t) = -t$. Since $\tilde{f}^{-1}(f(p_k)) = t_k$ and $\tilde{f}^{-1}(f(p_k)) \rightarrow \tilde{f}^{-1}(f(q))$, we have $t_k \rightarrow t_0 := \tilde{f}^{-1}(f(q))$. Letting $x := \gamma_q(-t_0) = \text{Exp}(-t_0\xi(q))$, we have

$$x_k = \gamma_{p_k}(-t_k) = \text{Exp}(-t_k\xi(p_k)) \rightarrow \gamma_q(-t_0).$$

Since N_0 is closed, x belongs to N_0 so that $q = F(t_0, x) \in M_0$. Thus $M = M_0$; furthermore, $N = N_0$.

Since M is homogeneous and T_2 is G -invariant, N is a homogeneous Riemannian manifold so that N is complete. Furthermore, N is simply connected, because M which is diffeomorphic to $\mathbf{R} \times N$, is simply connected. Therefore, in cases (i) and (iii) in Lemma 3, N is isometric to the Euclidean space $\mathbf{E}^{n-1}$. In case (ii) in Lemma 3, N is isometric to a sphere S^{n-1} or a hyperbolic space $\mathbf{H}^{n-1}$ by Lemma 4.

Let $(U, \phi = (t_1, \dots, t_{n-1}))$ be a local coordinate around a point p in N . Then $(\mathbf{R} \times U, \text{id} \times \phi = (t, t_1, \dots, t_{n-1}))$ is a local coordinate around (a, p) in $\mathbf{R} \times N$. Let $\tilde{U} := F(\mathbf{R} \times U)$ and define $\tilde{\phi}: \tilde{U} \rightarrow \mathbf{R}^n$ by $(\text{id} \times \phi) \circ F^{-1}$. Then $(\tilde{U}, \tilde{\phi} = (x_0, x_1, \dots, x_{n-1}))$ is a local coordinate around $\tilde{p} = F(a, p)$ in M . We can see that $dF(\partial/\partial t) = \xi = \partial/\partial x_0 \in T_1$ and $dF(\partial/\partial t_i) = \partial/\partial x_i \in T_2$ ($i = 1, \dots, n-1$). So we have $\langle \partial/\partial x_0, \partial/\partial x_0 \rangle = -1$ and $\langle \partial/\partial x_0, \partial/\partial x_i \rangle = 0$ for $1 \leq i \leq n-1$.

LEMMA 7. In cases (i) and (ii) in Lemma 3, $F: (\mathbf{R} \times N, -dt^2 + ds_N^2) \rightarrow (M, \langle \cdot, \cdot \rangle)$ is isometry, where ds_N^2 is the metric of N .

PROOF. It is enough to show that for $1 \leq i, j \leq n-1$, $\langle \partial/\partial x_i, \partial/\partial x_j \rangle$ is independent of x_0 . Since $\pi: (G, B) \rightarrow (M, \langle \cdot, \cdot \rangle)$ is the semi-Riemannian submersion, it follows from Lemma 3 that T_1 is parallel. So, for $1 \leq i, j \leq n-1$, we have

$$\begin{aligned} \langle \partial/\partial x_0, \langle \partial/\partial x_i, \partial/\partial x_j \rangle \rangle &= \langle \nabla_{\partial/\partial x_i} \partial/\partial x_0, \partial/\partial x_j \rangle \\ &\quad + \langle \partial/\partial x_i, \nabla_{\partial/\partial x_j} \partial/\partial x_0 \rangle \\ &= 0 \end{aligned}$$

because $\nabla_{\partial/\partial x_i} \partial/\partial x_0$ and $\nabla_{\partial/\partial x_j} \partial/\partial x_0$ belong to T_1 .

LEMMA 8. In case (iii) in Lemma 3, $F: (\mathbf{R} \times N, -dt^2 + \exp(-2ct)ds^2) \rightarrow (M, \langle \cdot, \cdot \rangle)$ is isometry, where ds^2 is the flat metric of $N = \mathbf{E}^{n-1}$.

PROOF. Since $\pi: (G, B) \rightarrow (M, \langle \cdot, \cdot \rangle)$ is the semi-Riemannian submersion, the equalities $\nabla_{\partial/\partial x_j} \partial/\partial x_0 = -c(\partial/\partial x_j)$ ($1 \leq j \leq n-1$) follows from Lemma 3. We have

$$(\partial/\partial x_0) \langle \partial/\partial x_i, \partial/\partial x_j \rangle = -2c \langle \partial/\partial x_i, \partial/\partial x_j \rangle$$

for $1 \leq i, j \leq n-1$ so that we have

$$\langle \partial/\partial x_i, \partial/\partial x_j \rangle = \exp(-2cx_0)g_{ij}(x_1, \dots, x_{n-1}).$$

Thus

$$F^* \langle \cdot, \cdot \rangle = -dt^2 + \exp(-2ct) \sum_{i,j=1}^{n-1} g_{ij} \circ F(t_1, \dots, t_{n-1}) dt_i dt_j.$$

Since N is isometric to $\mathbf{E}^{n-1}$, we may assume that $g_{ij} = \delta_{ij}$.

PROOF OF THEOREM. Consider case (i) in Lemma 3. If M is simply connected, M is isometric to $(\mathbf{R} \times \mathbf{E}^{n-1}, -dt^2 + ds^2)$ by Lemma 7 where ds^2 is the metric of $\mathbf{E}^{n-1}$. To find a non-simply-connected M , we must look for a discrete subgroup Γ of the full isometry group of $\mathbf{R} \times \mathbf{E}^{n-1}$ which commutes with $G = \mathbf{R} \times I^0(\mathbf{E}^{n-1})$ elementwise. It is easy to verify that Γ is generated by a translation of $\mathbf{R}$. Hence, if M is not simply connected, then M is isometric to $S^1(r) \times \mathbf{E}^{n-1}$.

Consider case (ii) in Lemma 3. If M is simply connected, M is isometric to $(\mathbf{R} \times N, -dt^2 + ds_N^2)$ by Lemma 7 where N is $S^{n-1}(r')$ or $\mathbf{H}^{n-1}(r')$. By the same method as above, if M is not simply connected, then M is isometric to $S^1(r) \times S^{n-1}(r')$, $\mathbf{R} \times \mathbf{P}^{n-1}$, $S^1(r) \times \mathbf{P}^{n-1}$ or $S^1(r) \times \mathbf{H}^{n-1}(r')$.

Consider case (iii) in Lemma 3. If M is simply connected, M is isometric to $(\mathbf{R} \times \mathbf{R}^{n-1}, -dt^2 + \exp(-2ct) \sum_{j=1}^{n-1} dt_j^2)$ by Lemma 8. This space is isometric to the space (v) in the Example by the transformation

$$\mathbf{R} \times \mathbf{R}^{n-1} \ni (t, t_1, \dots, t_{n-1}) \rightarrow (t_1, \dots, t_{n-1}, e^{ct}/c) \in U_n^+.$$

A discrete subgroup of the full isometry group $I(U_n^+)$ which commutes with $G = I^0(U_n^+)$ elementwise consists of the identity element.

The proof of the theorem is complete.

ACKNOWLEDGMENT. The author would like to thank Professor Y. Maeda for suggesting the problem. The author also would like to thank Professors H. Kitahara, S. Yorozu, and M. Obata for their helpful advice and encouragement, and the referee for his kind and suitable advice.

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