WU CLASSES

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ABSTRACT. This paper shows that a nonbounding closed manifold with Wu classes $v_i = 0$ for $i > k$ has dimension bounded by some function of k .

1. Introduction. Let M^n be a closed n -dimensional manifold. By Poincaré duality, there are unique classes $v_i \in H^i(M; Z_2)$ satisfying $v_i x[M] = \text{Sq}^i x[M]$ for all $x \in H^{n-i}(M; Z_2)$. Since $\text{Sq}^i x = 0$ for $i > n - i$, $v_i = 0$ for $i > [n/2]$, and the class

$$v = 1 + v_1 + v_2 + \cdots + v_{[n/2]} \in H^*(M; Z_2)$$

is called the total Wu class of M .

The Stiefel-Whitney class $w = 1 + w_1 + w_2 + \cdots + w_n$ of M is given by $w = \text{Sq} v$, where $\text{Sq} = 1 + \text{Sq}^1 + \text{Sq}^2 + \cdots$, or equivalently $w_i = \sum_{j=0}^i \text{Sq}^{i-j} v_j$. It is well known that the Stiefel-Whitney numbers $w_{i_1} \cdots w_{i_r}[M]$ determine the cobordism class of M . In particular, if $n > 0$ and M is not a boundary, there must be an $i > 0$ for which $v_i \neq 0$.

The purpose of this note is to show that $v_i \neq 0$ for some moderately large value of i , and specifically to prove the following proposition.

PROPOSITION. *For any integer k there is an integer $N = N(k)$ so that a closed manifold M with Wu class $v = 1 + v_1 + \cdots + v_k$ bounds if $\dim M > N$.*

No analogous result holds if one considers other characteristic classes. In particular, one has the following.

OBSERVATION. *There is a nonbounding closed manifold of dimension 2^{r+1} with Stiefel-Whitney class $w = 1 + w_1 + w_2$.*

OBSERVATION. *There is a nonbounding closed manifold of dimension $2^s - 2$ with normal Stiefel-Whitney class $\bar{w} = 1/w = 1 + \bar{w}_1$.*

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2. The proof. In order to prove the proposition, one makes use of the space BO which classifies stable vector bundles. One then has $H^*(BO; Z_2) = Z_2[w_i | i \geq 1]$, where w_i is the universal Stiefel-Whitney class. One has

LEMMA. *Let $B_k^* = H^*(BO; Z_2)/I(v_i | i > k)$, where $I(v_i | i > k)$ is the ideal generated over the Steenrod algebra by the classes v_i , $i > k$, where $v = \text{Sq}^{-1} w$. Then B_k^* is finite dimensional.*

The proposition is then immediate by taking $N(k)$ to be the largest integer i for which $B_k^i \neq 0$.

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To prove the lemma, one may proceed by induction on k . For $k = 1$, B_1^* has $v = 1 + v_1$ and $w = \text{Sq } v = 1 + v_1 + v_1^2$ with

$$0 = \text{Sq}^1(v_1^2) = \text{Sq}^1 w_2 = w_3 + w_2 w_1 = w_2 w_1 = v_1^3.$$

Thus $B_1^* = Z_2 \oplus Z_2 \oplus Z_2$ with basis $1, v_1, v_1^2$. One then assumes B_j^* is finite dimensional for all $j < k$, and considers B_k^* in which $v = 1 + v_1 + \cdots + v_k$ and

$$w = 1 + v_1 + (v_2 + v_1^2) + \cdots + (\text{Sq}^{k-1} v_k) + (v_k^2).$$

Step 1. There is an integer r for which $v_k^{2^r} = 0$. For k odd, one has

$$\text{Sq}^1 w_{2k-1} = w_1 w_{2k-1}$$

but

$$\text{Sq}^1 w_{2k-1} = \text{Sq}^1 \text{Sq}^{k-1} v_k = \text{Sq}^k v_k = v_k^2 = w_{2k}.$$

Also

$$\text{Sq}^1 w_{2k} = \text{Sq}^1(v_k^2) = 0$$

but

$$\text{Sq}^1 w_{2k} = w_{2k+1} + w_1 w_{2k} = w_1 w_{2k}.$$

Thus

$$v_k^4 = w_{2k} w_{2k} = w_{2k} w_1 w_{2k-1} = 0 \cdot w_{2k-1} = 0.$$

Now suppose k is even. One then has

$$0 = \text{Sq}^{2j+1}(v_k^2) = \text{Sq}^{2j+1} w_{2k} = w_{2j+1} w_{2k} = w_{2j+1} v_k^2.$$

Let $I_r = \{\alpha \in B_k^* \mid \alpha v_k^{2^r} = 0\}$. One then has $I_0 \subset I_1 \subset I_2 \subset \cdots$ and lets $I = \bigcup_{r=0}^{\infty} I_r$.

Claim. I is an ideal closed under the action of the Steenrod algebra.

PROOF. If $\alpha \in I$, $\beta \in B_k^*$, there is an r with $\alpha \in I_r$ so $\alpha v_k^{2^r} = 0$. Then $\alpha \beta v_k^{2^r} = 0$ so $\alpha \beta \in I_r$ giving $\alpha \beta \in I$. If $\alpha \in I$ and j is an integer, one wants to see that $\text{Sq}^j \alpha \in I$. Since $\alpha \in I$, there is an r with $\alpha \in I_r$ or $\alpha v_k^{2^r} = 0$. Letting $s > r$ with $2^s > j$ one has

$$\text{Sq}^j \alpha \cdot v_k^{2^s} = \text{Sq}^j(\alpha v_k^{2^s}) = \text{Sq}^j(\alpha v_k^{2^r} \cdot v_k^{2^s-2^r}) = \text{Sq}^j(0) = 0,$$

so $\text{Sq}^j \alpha \in I_s$ and $\text{Sq}^j \alpha \in I$. $\square$

One now considers B_k^*/I in which $w_{2q+1} = 0$ since, $w_{2q+1} \in I_1 \subset I$. As is well known [M], $H^*(BO; Z_2)$ modulo the ideal generated by the odd w_i is isomorphic to $H^*(BU; Z_2)$, which is isomorphic to $H^*(BO; Z_2)$, but with all dimensions formally doubled. Thus, one has a homomorphism

$$B_{k/2}^* \rightarrow B_k^*/I: w_i \rightarrow w_{2i}$$

which is epic. Inductively, $B_{k/2}^*$ is finite dimensional, so B_k^*/I is finite dimensional.

Thus there is an s with $v_k^{2^s} \in I$, and so $v_k^{2^s} \in I_r$ for some r , or $v_k^{2^s} \cdot v_k^{2^r} = 0$. There is then a q with $v_k^{2^q} = 0$, completing the proof of Step 1.

Step 2. There is an integer q so that $x^{2^q} = 0$ for all $x \in B_k^$, $x \neq 1$.*

To see this, one chooses r from Step 1 so that $v_k^{2^r} = 0$ and considers

$$\phi: B_k^* \rightarrow B_k^*: x \rightarrow x^{2^r}.$$

The kernel of this homomorphism is an ideal over the Steenrod algebra. If $\phi(x) = 0$, then $\phi(xy) = \phi(x)\phi(y) = 0$ and $\phi(\text{Sq}^i x) = (\text{Sq}^i x)^{2^r} = \text{Sq}^{2^r i}(x^{2^r}) = \text{Sq}^{2^r i}(0) = 0$. Since $\phi(v_k) = 0$, ϕ factors through B_{k-1}^* , and by induction B_{k-1}^* is finite dimensional, so $\phi(B_k^*)$ is finite dimensional. Thus, there is an m so that $\phi(B_k^*)^i = 0$ if $i > m$ and in particular, there is an s with $2^s > m$, so that for any $x \in B_k^*$, $x^{2^{r+s}} = \phi(x^{2^s}) = 0$ being in image ϕ and having dimension larger than m . Taking $q = r + s$ gives the desired assertion.

Finally, B_k^* is generated by $w_1, w_2, \dots, w_{2k}$ and has $w_i^{2^q} = 0$ for each i , so $B_k^j = 0$ if $j \geq 2^q(1 + 2 + \dots + 2k) = 2^q \binom{2k+1}{2}$. Thus B_k^* is finite dimensional.

3. Comments. *Comment 1.* The manifold RP^{2^r-2} is nonbounding and has $\bar{w} = 1 + \bar{w}_1$, so no analogous result holds for the normal Stiefel-Whitney class.

Comment 2. The manifold $M^{2^{r+1}} = RP(\xi_1 \oplus \xi_2 \oplus 1)$ fibered over $RP^{2^r-1} \times RP^{2^r-1}$ with fiber RP^2 , ξ being the nontrivial line bundle over the i th factor, is nonbounding and has $w = 1 + w_1 + w_2$. Hence, no analogous statement holds for the Stiefel-Whitney class.

Comment 3. It would be of interest to find the smallest value of $N(k)$ which has the property described in the proposition. One has a prospect of tracing through the argument to find a bound, but that will be too large. In fact, $N(k)$ can be less than the largest i for which $B_k^i \neq 0$.

Unpleasant calculations give $N(1) = 2$, $N(2) = 5$, $N(3) = 7$, which coincides with $3k - \alpha(k)$, where α is the number of one's in the 2-adic expansion. That bound does not work generally (seemingly $N(6) \geq 19$).

Comment 4. One potential application is given by

PROPOSITION. *If $N(k) \leq 4k$ for $k \leq [(j-1)/2]$, then every closed M^n which fibers over the sphere S^j with $n < 2j$ is a boundary.*

PROOF. If $F^{n-j} \xrightarrow{i} M^n \xrightarrow{\pi} S^j$ is a fibering with $n < 2j$, then $v_i(M) = 0$ for $i > [(n-j)/2]$. $\square$

Note. In [D], it is shown that if M^n fibers over S^j and $n < (3j-2)/2$, then M bounds (see Proposition 2.4).

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