RATIONAL CURVE WITH ONE CUSP. II

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ABSTRACT. Let C be a plane curve with $C - \{P\} \cong \mathbf{A}^1$ for some point $P \in C$ and degree $d \geq 3$. Let $\{e_1, \dots, e_t\}$ be the multiplicities of the infinitely near singular points of P . Then the following three conditions are equivalent:

- (1) $C \setminus L \cong \mathbf{A}^1$ for some line L ,
- (2) $d = e_1 + e_2$ (in case $t = 1$, let $e_2 = 1$),
- (3) $R = d^2 - \sum_{i=1}^t e_i^2 - e_t + 1 \geq 3$.

1. This paper is a continuation of [2]. We shall give an affirmative answer to the problem raised in that paper, which is a numerical criterion for a curve of type I to be type II. By the criterion the relation between types I and II will be clear. We follow the notation and convention used in [2].

THEOREM. *Suppose that C is a curve of type I and $d \geq 3$. Then C is of type II if and only if $R \geq 3$.*

Recalling the result [2, Lemma 1], we see that conditions (1) and (2) in the abstract are equivalent. In case $d = 1$ and 2, since $t = 0$, we put $R = 2$ and 5 respectively. Then we obtain the following from the lemma in the next section.

COROLLARY. *Suppose that C is a curve of type I. Then it is a line if and only if $R = 2$.*

2. We shall prove the next lemma. Then the proof of the Theorem will be complete, since the only if part has been proved [3, Lemma 5.1].

LEMMA. *If C is a curve of type I with $R \geq 2$, then it is of type II.*

PROOF. First we fix some notation. Let

$$\mathbf{P}^2 = S_0 \xleftarrow{\sigma_1} S_1 \xleftarrow{\sigma_2} \cdots \xleftarrow{\sigma_{k-1}} S_{k-1} \xleftarrow{\sigma_k} S_k$$

be a sequence of blow-ups σ_i with successive centers P_i in S_{i-1} and put $\sigma = \sigma_1 \cdots \sigma_k$. For a birational transformation g we denote by $g(X)$ and $g[X]$ the total and proper transforms of X respectively. Then put $E_i = \sigma_i^{-1}(P_i)$ and $E'_i = (\sigma_{i+1} \cdots \sigma_k)^{-1}[E_i]$, where $1 \leq i \leq k-1$. For a curve X on a surface, let X^2 denote the self-intersection number of X on the surface. Let k be minimal in order that $\sigma^{-1}(C)$ has only simple normal crossings and $C'^2 = 0$, where $C' = \sigma^{-1}[C]$. This is possible, since $R \geq 2$. Then we have $k = t + e_t + R - 1$. Since C' is a nonsingular rational curve with $C'^2 = 0$, there is a morphism $f: S = S_k \rightarrow \mathbf{P}^1$ defined by the linear system $|C'|$ [1]. Putting $\mathcal{E} = E'_1 + \cdots + E'_{k-1}$, we have that $\mathcal{E} \cap C' = \emptyset$. Since $R \geq 2$, $\mathcal{E}$ is connected and hence it is contained in a single fiber $\mathcal{F}$. By the above definition we see that E_k is a section of f and $E_k^2 = -1$.

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Note that $E_i'^2 \leq -2$ for $i \neq k$. Referring to [1] again, we see that $\mathcal{F}$ contains an exceptional curve of the first kind Δ' and $\mathcal{F} = \mathcal{E} + \Delta'$ as a divisor. Comparing the number of blow-ups to obtain S from $\mathbf{P}^2$ with that of the irreducible components of $\mathcal{F}$, we infer that f has no singular fiber except $\mathcal{F}$. By contracting successively the components of $\mathcal{F}$, we get a birational morphism $\tau: S \rightarrow F$ and then a morphism $\bar{f}: F \rightarrow \mathbf{P}^1$, where $\bar{f}$ has no singular fiber. Here we may assume that E_{k-1}' is not contracted by τ . Otherwise τ^{-1} has fundamental point at $\tau(E_k \cap \mathcal{F})$. Then by elementary transformation with the center at this point we have a new surface which S dominates, too. We can repeat this procedure to get the surface on which the self-intersection of the image of E_k becomes -1 , as was to be shown. Then, by contracting E_k , we get an isomorphism $\mathbf{P}^2 - \Delta \cong \mathbf{P}^2 - L$, where $\Delta = \sigma(\Delta')$ and L is a line. Since the group $\text{Pic}(\mathbf{P}^2 - \Delta)$ is a cyclic group whose order is the degree Δ , the curve Δ is also a line. On the other hand, by the above definition, we have that $C \cap \Delta = \{P\}$. This means that C is of type II.

REFERENCES

1. M. H. Gizatullin, *On affine surfaces that can be completed by a nonsingular rational curve*, Math. USSR-Izv. **4** (1970), 787–810.
2. H. Yoshihara, *Rational curve with one cusp*, Proc. Amer. Math. Soc. **89** (1983), 24–26.
3. ———, *On open algebraic surfaces $\mathbf{P}^2 - C$* , Math. Ann. **268** (1984), 43–57.

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