A WEAK GAGA STATEMENT FOR ARBITRARY MORPHISMS

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ABSTRACT. Let $f: X \rightarrow Y$ be an arbitrary morphism of schemes of finite type over $\mathbb{C}$, and let f^{an} be the associated map of analytic spaces. Let $\mathcal{S}$ be a coherent sheaf on X . Then $(f_*\mathcal{S})^{\text{an}} \rightarrow f_*^{\text{an}}(\mathcal{S}^{\text{an}})$ is injective.

0. Introduction. Serre's paper GAGA, as generalized by Gorthendieck, proves the following result. Let X and Y be schemes of finite type over $\mathbb{C}$. Let $f: X \rightarrow Y$ be a proper morphism. Let $\mathcal{S}$ be a coherent sheaf on X . Then if $^{\text{an}}$ stands for the analytification functor, the natural map $\phi: (R^i f_* \mathcal{S})^{\text{an}} \rightarrow R^i f_*^{\text{an}}(\mathcal{S}^{\text{an}})$ is an isomorphism. It is well known that this breaks down if f is not proper.

In this note, I prove that one can rescue an infinitesimal amount in the general case. If $f: X \rightarrow Y$ is arbitrary, I prove that $(f_* \mathcal{S})^{\text{an}} \rightarrow f_*^{\text{an}} \mathcal{S}^{\text{an}}$ is injective.

One application of this result is to give an alternative approach to my proof of injectivity in Neeman [N, Lemmas 4.7 and 4.8 and Proposition 4.9]. One way to view the original proof is as a trick that gets around the theory developed here.

1. Preliminaries. In this section I want to review some of the definitions in Serre's paper (Serre [GAGA]). Let X be a scheme of finite type over $\mathbb{C}$. Serre defined what is meant by the analytification X^{an} of X (Serre denotes that space X^h). He also defined what is meant by the analytification of an algebraic sheaf on X . Allow me to recall the definitions so as to remind the reader of the basic setup.

There is an underlying continuous map of ringed topological spaces $X^{\text{an}} \rightarrow X$. Let $\mathcal{S}$ be a sheaf of $\mathcal{O}_X$ -modules on X . The analytification $\mathcal{S}^{\text{an}}$ of $\mathcal{S}$ is defined to be the sheaf $\phi^{-1} \mathcal{S} \otimes_{\phi^{-1} \mathcal{O}_X} \mathcal{O}_{X^{\text{an}}}$.

REMARK 1.1. $\mathcal{S}^{\text{an}}$ is a sheaf of $\mathcal{O}_{X^{\text{an}}}$ -modules on X^{an} . The map $\mathcal{S} \mapsto \mathcal{S}^{\text{an}}$ extends to a functor from the category of $\mathcal{O}_X$ -modules over X to the category of $\mathcal{O}_{X^{\text{an}}}$ -modules on X^{an} . We also denote this functor by ϕ^* .

REMARK 1.2. The functor ϕ^* is left adjoint to the functor ϕ_* (viewed as a functor from the category of $\mathcal{O}_{X^{\text{an}}}$ -modules to the category of $\mathcal{O}_X$ -modules).

REMARK 1.3. The stalk of $\mathcal{S}^{\text{an}}$ at a point $x \in X^{\text{an}}$ is given by the formula $\mathcal{S}_x^{\text{an}} = \mathcal{S}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X^{\text{an}},x}$.

REMARK 1.4. Analytification is an exact functor. This can be checked on stalks, so it follows from Remark 1.3 (cf. Serre [GAGA, §3, Proposition 10]).

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Let $f: X \rightarrow Y$ be a morphism of schemes. One defines $f^{\text{an}}: X^{\text{an}} \rightarrow Y^{\text{an}}$, and easily verifies that the following diagram of continuous maps of ringed topological spaces commutes:

$$\begin{array}{ccc} X^{\text{an}} & \xrightarrow{\phi_X} & X \\ f^{\text{an}} \downarrow & & \downarrow f \\ Y^{\text{an}} & \xrightarrow{\phi_Y} & Y \end{array}$$

On X , we have a natural map $f^* f_* \mathcal{S} \rightarrow \mathcal{S}$. We deduce a map $\phi_X^* f^* f_* \mathcal{S} \rightarrow \phi_X^* \mathcal{S}$. By the commutativity of the diagram, there is a natural isomorphism $\phi_X^* f^* \cong (f^{\text{an}})^* \phi_Y^*$. So we deduce a map $(f^{\text{an}})^* \phi_Y^* f_* \mathcal{S} \rightarrow \phi_X^* \mathcal{S}$. Finally, we get a map by adjunction $\phi_Y^* f_* \mathcal{S} \rightarrow f_*^{\text{an}} \phi_X^* \mathcal{S}$. In other words, there is a natural map $\phi: (f_* \mathcal{S})^{\text{an}} \rightarrow f_*^{\text{an}}(\mathcal{S}^{\text{an}})$.

REMARK 1.5. A fact that we will not need in this paper is that this natural comparison map can be extended to compare the higher derived functors of f_* . Precisely, observe that because analytification is exact, the functors $\mathcal{S} \mapsto (R^i f_* \mathcal{S})^{\text{an}}$ are the higher derived functors of $\mathcal{S} \mapsto (f_* \mathcal{S})^{\text{an}}$. Because $\mathcal{S} \mapsto R^i f_*^{\text{an}}(\mathcal{S}^{\text{an}})$ is also a δ -functor (again, we use the exactness of analytification), it follows that there is a unique natural comparison map of δ -functors $(R^i f_* \mathcal{S})^{\text{an}} \rightarrow R^i f_*^{\text{an}}(\mathcal{S}^{\text{an}})$.

REMARK 1.6. If $\mathcal{S}$ is a coherent sheaf on X , $\mathcal{S}^{\text{an}}$ is a coherent sheaf on X^{an} .

2. The weak GAGA theorem. We begin by proving

LEMMA 2.1. *Let X be an affine scheme of finite type over $\mathbb{C}$. Let $\mathcal{S}$ be a coherent algebraic sheaf on X . Let U be a Stein open subset of X^{an} . Then the natural map $\Gamma(X, \mathcal{S}) \otimes_{\Gamma(X, \mathcal{O}_X)} \Gamma(U, \mathcal{O}_{X^{\text{an}}}) \rightarrow \Gamma(U, \mathcal{S}^{\text{an}})$ is an isomorphism.*

PROOF. The lemma is clear if $\mathcal{S} \cong \mathcal{O}_X$. But because $\mathcal{S}$ is coherent on the affine scheme X , it admits a resolution $\mathcal{O}_X^m \rightarrow \mathcal{O}_X^n \rightarrow \mathcal{S} \rightarrow 0$. Analytification is exact, and because U is Stein, the functor sending coherent sheaves on U to their global sections is also exact. It follows that the rows in the commutative diagram below are exact:

$$\begin{array}{ccccccc} \Gamma(X, \mathcal{O}_X^m) \otimes_{\Gamma(X, \mathcal{O}_X)} \Gamma(U, \mathcal{O}_{X^{\text{an}}}) & \rightarrow & \Gamma(U, \mathcal{O}_X^m) \otimes_{\Gamma(X, \mathcal{O}_X)} \Gamma(U, \mathcal{O}_{X^{\text{an}}}) & \rightarrow & \Gamma(X, \mathcal{S}) \otimes_{\Gamma(X, \mathcal{O}_X)} \Gamma(U, \mathcal{O}_{X^{\text{an}}}) & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ & & \Gamma(U, (\mathcal{O}_X^{\text{an}})^m) & \rightarrow & \Gamma(U, (\mathcal{O}_X^{\text{an}})^n) & \rightarrow & \Gamma(U, \mathcal{S}^{\text{an}}) \rightarrow 0 \end{array}$$

From this the lemma is immediate. Q.E.D.

Next we prove

LEMMA 2.2. *Let X and Y be affine schemes of finite type over $\mathbb{C}$. Then, for any $y \in Y$, the ring*

$$\lim_{\substack{\longrightarrow \\ U \text{ open in } Y^{\text{an}} \\ y \in U}} \Gamma(X^{\text{an}} \times U, \mathcal{O}_{X^{\text{an}} \times Y^{\text{an}}})$$

is faithfully flat over $\Gamma(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \mathcal{O}_{Y^{\text{an}}, y}$.

PROOF. First we prove flatness. Let I be an ideal of $\Gamma(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \mathcal{O}_{Y^{\text{an}}, y}$. $\Gamma(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \mathcal{O}_{Y^{\text{an}}, y}$ is noetherian (it is finitely generated algebra over $\mathcal{O}_{Y^{\text{an}}, y}$), so I is a finitely presented ideal. Now we want to stretch this to a little neighborhood of y in Y^{an} .

Let $\bar{X}$ be a projective scheme over $\mathbb{C}$ such that $X \subset \bar{X}$ is an open immersion and $\bar{X} - X$ is the vanishing of one section $w \in \Gamma(\bar{X}, \mathcal{O}(1))$ (i.e. compactify X to $\bar{X}$ by adding a divisor). Then the ideal I in $\Gamma(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \mathcal{O}_{Y^{\text{an}}, y}$ extends to an ideal sheaf $\mathcal{I}$ on $\bar{X} \times_{\text{Spec } \mathbb{C}} \text{Spec } \mathcal{O}_{Y^{\text{an}}, y}$. We can resolve the ideal sheaf $\mathcal{I}$: precisely, we get a resolution $\mathcal{O}(-n_1)^{r_1} \rightarrow \mathcal{O}(-n_2)^{r_2} \rightarrow \mathcal{I}$ of $\mathcal{I}$. Because $\bar{X}^{\text{an}}$ is compact, we can stretch the resolution a little in the Y -direction; there exists $U \subset Y$ open, an ideal sheaf $\mathcal{I}^{\text{an}}$ on $\bar{X}^{\text{an}} \times U$ and a resolution $\mathcal{O}(-n_1)^{r_1} \rightarrow \mathcal{O}(-n_2)^{r_2} \rightarrow \mathcal{I}^{\text{an}}$. If we extend I to $\Gamma(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Gamma(U, \mathcal{O}_{Y^{\text{an}}})$ by defining it to be the quotient of the corresponding map $\Gamma(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Gamma(U, \mathcal{O}_{Y^{\text{an}}})^{r_1} \rightarrow \Gamma(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Gamma(U, \mathcal{O}_{Y^{\text{an}}})^{r_2}$, then by an argument exactly parallel to that of Lemma 2.1, we deduce that, provided U is a Stein space, the natural map

$$I \otimes_{\Gamma(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Gamma(U, \mathcal{O}_{Y^{\text{an}}})} \Gamma(X^{\text{an}} \times U, \mathcal{O}_{X^{\text{an}} \times Y^{\text{an}}}) \rightarrow \Gamma(X^{\text{an}} \times U, \mathcal{I}^{\text{an}})$$

is an isomorphism. But the map $\Gamma(X^{\text{an}} \times U, \mathcal{I}^{\text{an}}) \rightarrow \Gamma(X^{\text{an}} \times U, \mathcal{O}_{X^{\text{an}} \times Y^{\text{an}}})$ is injective. Shrinking U and taking direct limits, the flatness we asserted follows.

Next, we prove faithful flatness. Suppose $f_1, \dots, f_r \in \Gamma(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \mathcal{O}_{Y^{\text{an}}, y}$ generate the unit ideal in $\lim_{\rightarrow} \Gamma(X^{\text{an}} \times U, \mathcal{O}_{X^{\text{an}} \times Y^{\text{an}}})$. We have to show that they already generate the unit ideal in $\Gamma(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \mathcal{O}_{Y^{\text{an}}, y}$.

For small enough U , $f_1, \dots, f_r$ already lift to elements of $\Gamma(X^{\text{an}} \times U, \mathcal{O}_{X^{\text{an}}, Y^{\text{an}}})$. Shrinking U , we may assume that they generate the unit ideal in $\Gamma(X^{\text{an}} \times U, \mathcal{O}_{X^{\text{an}} \times Y^{\text{an}}})$. This means that the functions $f_1, \dots, f_r$ have no common zeros on $X^{\text{an}} \times U$.

We compactify X to $\bar{X}$ as before. For some n sufficiently large, $f_1, \dots, f_r$ extend to sections of $\Gamma(\bar{X}^{\text{an}} \times U, \mathcal{O}(n)^{\text{an}})$, at least if we shrink U suitably (unless we shrink U , the degree of the f_i 's may become unbounded).

The fact that $f_1, \dots, f_r$ have no common zero on $X^{\text{an}} \times U$ can be restated as saying that the cokernel of the corresponding map $[\mathcal{O}(-n)^{\text{an}}]^r \rightarrow \mathcal{O}^{\text{an}}$ is supported on $(\bar{X} - X)^{\text{an}} \times U$. Shrinking U some more, the cokernel will be annihilated by w^m for some $m > 0$.

Twisting by a large N and shrinking U some more to a small Stein space, we can ensure that the map $\Gamma(\bar{X}^{\text{an}} \times U, [\mathcal{O}(-n + N)^{\text{an}}]^r) \rightarrow \Gamma(\bar{X}^{\text{an}} \times U, \mathcal{O}(N)^{\text{an}})$ will surject to the kernel of the map to the quotient sheaf. If N is chosen $\geq m$, w^N lies in the image.

Reading this on $X \subset \bar{X}$, we see that we have expressed 1 as a $\Gamma(X, \mathcal{O}_X) \otimes_{\mathbb{C}} \Gamma(U, \mathcal{O}_{Y^{\text{an}}})$ linear combination of the f_i 's.

(I have slurred over several points here. For instance, we are using the usual GAGA applied to the projective space $\bar{X}$, to deduce that $\Gamma(\bar{X}, \mathcal{O}(p)) \rightarrow \Gamma(\bar{X}^{\text{an}}, \mathcal{O}(p)^{\text{an}})$ is an isomorphism. We are actually using a slight generalization; we need to do this on $\bar{X} \times U$. I leave it to the reader to complete the proof in these details.) Q.E.D.

LEMMA 2.3. *Let X and Y be affine schemes of finite type over $\mathbf{C}$. Let $\pi: X \times Y \rightarrow Y$ be the projection. If $\mathcal{S}$ is a coherent sheaf on $X \times Y$, $\phi: (\pi_* \mathcal{S})^{\text{an}} \rightarrow \pi_*^{\text{an}}(\mathcal{S}^{\text{an}})$ is injective.*

PROOF. This can be studied on stalks at $y \in Y$. The stalk of the first sheaf is

$$\begin{aligned} \lim_{\substack{\longrightarrow \\ y \in U \subset Y^{\text{an}}}} \Gamma(X \times Y, \mathcal{S}) \otimes_{\Gamma(Y, \mathcal{O}_Y)} \Gamma(U, \mathcal{O}_{Y^{\text{an}}}) \\ \cong \Gamma(X \times Y, \mathcal{S}) \otimes_{\Gamma(X, \mathcal{O}_X) \otimes_{\mathbf{C}} \Gamma(Y, \mathcal{O}_Y)} \Gamma(X, \mathcal{O}_X) \otimes_{\mathbf{C}} \mathcal{O}_{Y^{\text{an}}, y}. \end{aligned}$$

The stalk of the second sheaf is

$$\begin{aligned} \lim_{\substack{\longrightarrow \\ y \in U \subset Y^{\text{an}}}} \Gamma(X^{\text{an}} \times U, \mathcal{S}) \\ \cong \lim_{\substack{\longrightarrow \\ y \in U \subset Y}} \Gamma(X \times Y, \mathcal{S}) \otimes_{\Gamma(X, \mathcal{O}_X) \otimes_{\mathbf{C}} \Gamma(Y, \mathcal{O}_Y)} \Gamma(X^{\text{an}} \times U, \mathcal{O}_{X^{\text{an}} \times Y^{\text{an}}}) \end{aligned}$$

(the isomorphism is a consequence of Lemma 2.1). The natural map is injective by the faithful flatness assertion of Lemma 2.2. Q.E.D.

LEMMA 2.4. *Let $f: X \rightarrow Y$ be a morphism of affine schemes of finite type over $\mathbf{C}$. Let $\mathcal{S}$ be a coherent sheaf on X . Then $\phi: (f_* \mathcal{S})^{\text{an}} \rightarrow f_*^{\text{an}}(\mathcal{S}^{\text{an}})$ is injective.*

PROOF. Express f as the composite of a closed immersion and a projection (i.e. $X \xrightarrow{i} X \times Y \xrightarrow{\pi} Y$, the first map being the graph). The ϕ is the composite $(\pi_* i_* \mathcal{S})^{\text{an}} \rightarrow \pi_*^{\text{an}}(i_* \mathcal{S})^{\text{an}} \rightarrow \pi_*^{\text{an}} i_*^{\text{an}} \mathcal{S}^{\text{an}}$. The first map is injective by Lemma 2.3, the second is an isomorphism (analytification commutes with closed immersions). Q.E.D.

Finally, we prove

THEOREM 2.5. *Let $f: X \rightarrow Y$ be an arbitrary morphism of schemes of finite type over $\mathbf{C}$. Let $\mathcal{S}$ be a coherent sheaf on X . Then $\phi: (f_* \mathcal{S})^{\text{an}} \rightarrow f_*^{\text{an}}(\mathcal{S}^{\text{an}})$ is injective.*

PROOF. The problem is local in Y , so we may assume Y affine. Cover X by affines X_i , $i \in I$, where I is a finite set. Then if $f|_{X_i} = f_i$, $[(f_i)_* \mathcal{S}]^{\text{an}} \rightarrow (f_i^{\text{an}})_* \mathcal{S}^{\text{an}}$ is injective by Lemma 2.4. Hence $\prod_{i \in I} [(f_i)_* \mathcal{S}]^{\text{an}} \rightarrow \prod_{i \in I} (f_i^{\text{an}})_* (\mathcal{S}^{\text{an}})$ is injective. It follows that $\phi: (f_* \mathcal{S})^{\text{an}} \rightarrow f_*^{\text{an}}(\mathcal{S}^{\text{an}})$ is an injection, because $(f_* \mathcal{S})^{\text{an}}$ is a subsheaf of $\prod_{i \in I} [(f_i)_* \mathcal{S}]^{\text{an}}$, while $f_*^{\text{an}} \mathcal{S}^{\text{an}}$ is a subsheaf of $\prod_{i \in I} (f_i^{\text{an}})_* (\mathcal{S}^{\text{an}})$. Q.E.D.

ADDED IN PROOF. It has been pointed out to me that Lemma 2.1 may be found in Schwarz [S, §6]. What is new here is Lemma 2.2, and its application.

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