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POLYNOMIALS OF AN INNER FUNCTION WHICH ARE EXPOSED POINTS IN H^1

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ABSTRACT. It is known that if $p(z)$ is an analytic polynomial which has no zeros in the open unit disc and distinct zeros in the unit circle, then $p(z)/\|p(z)\|_1$ is an exposed point of the unit ball of the Hardy space H^1.

In this paper, it is proved that for a bounded analytic function f with $\|f\|_\infty \leqslant 1$, $p(f)/\|p(f)\|_1$ is also an exposed point.

Let U be the open unit disc in the complex plane and let ∂U be the boundary of U. If f is analytic in U and $\int_{-\pi}^{\pi} \log|f(re^{i\theta})|\,d\theta$ is bounded for $0 \leqslant r < 1$, then $f(e^{i\theta})$, which we define to be $\lim_{r\to 1} f(re^{i\theta})$, exists almost everywhere on ∂U. If

$$\lim_{r\to 1} \int_{-\pi}^{\pi} \log^+|f(re^{i\theta})|\,d\theta = \int_{-\pi}^{\pi} \log^+|f(e^{i\theta})|\,d\theta,$$

then f is said to be in the class N_+. The set of all boundary functions in N_+ is denoted by N_+ again. For $0 < p \leqslant \infty$, the Hardy space H^p is defined by $N_+ \cap L^p$. A denotes the disc algebra, that is $A = \{f\colon f$ is continuous on $\overline{U}$ and analytic in $U\}$. If h in N_+ has the form

$$h(z) = \exp\left\{\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z}\,\frac{\log|h(e^{it})|}{2\pi}\,dt + i\alpha\right\}$$

for some real α, h is called an outer function. We call q in N_+ an inner function if $|q(e^{i\theta})| = 1$ a.e. on ∂U.

Let g be a nonzero function in H^p. Then the following property $(*)$ characterizes that g is an outer function.

$(*)$ Whenever kg belongs to H^p for k in L^∞ with $k(e^{i\theta}) \geqslant 0$ a.e. on U, then k is a constant function (see **[6]**).

We can consider a stronger property of g:

$(**)$ Whenever kg belongs to H^P for some Lebesgue measurable k with $k(e^{i\theta}) \geqslant 0$ a.e. on ∂U, then k is a constant function.

In **[6]**, the function g with property $(**)$ is called a p-strong outer function. We should remark that deLeeuw and Rudin **[1]** used the phrase "strong outer function" in a little different context. The p-strong outer functions appear to be important in many problems, for example, extremal problems, interpolation problems, Toeplitz

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operators, and prediction theory. In particular, when $\|g\|_1 = 1$, g is a 1-strong outer function if and only if g is exposed points of the unit ball of H^1 (see [**6**]).

Suppose $p(z) = \prod_{j=1}^n (z + a_j)$. If $p(z)/\|p(z)\|_1$ is an exposed point, then $|a_j| \geqslant 1$ $(j = 1, \ldots, n)$ and $a_i \neq a_j$ $(i \neq j)$ (cf. [**1**]). It is known that the converse is valid [**7**], which is also derived from [**4** and **5**] as follows. If $n = 1$, the result follows from Proposition 5 of [**4**]. Suppose $n > 1$ and $\prod_{i=1}^{n-1}(z + a_j)/\|\prod_{j=1}^{n-1}(z + a_j)\|_1$ is exposed but $p(z)/\|p(z)\|_1$ is not. Here we may assume without loss of generality that $|a_j| = 1$ for some j, say $j = n$. By Proposition 1 of [**4**], we have an element k in $S^0_{|p|/p}$ which can be represented by $(e^{i\theta} + a_n)(1 + \bar{a}_n e^{i\theta})h(e^{i\theta})$ for some nonconstant h in H^1 by Lemma 3 of [**5**]. Thus we have

$$k/p = (1 + \bar{a}_n z)h/\prod_{j=1}^{n-1}(z + a_j) \geqslant 0 \quad \text{a.e. on } \partial U,$$

which contradicts the assumption that $\prod_{j=1}^{n-1}(z + a_j)/\|\prod_{j=1}^{n-1}(z + a_j)\|_1$ is exposed.

Now we wish to prove that $p(f)/\|p(f)\|_1$ is an exposed point for the above $p(z)$ and any nonconstant f in H^∞ with $\|f\|_\infty \leqslant 1$. For $n = 1$, this is known [**4**, Proposition 5]. But we need a new idea to prove it in general.

LEMMA. *If $P(z) = \prod_{j=1}^n (z + a_j)$, $|a_j| = 1$ $(j = 1, \ldots, n)$, and $a_i \neq a_j$ $(i \neq j)$, then there exists a k in A such that k^{-1} is in A and* $\operatorname{Re}[k(e^{i\theta})p(e^{i\theta})] \geqslant 0$ *a.e. on ∂U.*

PROOF. By the hypothesis on a_j, we can write $a_j = e^{i(\alpha_j - \pi)}$ $(j = 1, \ldots, n)$, where $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n \leqslant 2\pi$. Let

$$s = \left[\frac{\sum_{j=1}^n (\alpha_j - \pi)}{4\pi}\right] \quad \text{and} \quad \alpha = 2\pi\left(\frac{\sum_{j=1}^n (\alpha_j - \pi)}{4\pi} - s\right),$$

where $[\cdot]$ is the greatest integer function, and we have $0 \leqslant \alpha < 2\pi$. Then there exists a real valued function $\nu(\theta)$ on $[0, 2\pi]$ such that

(i) $p(e^{i\theta})/|p(e^{i\theta})| = e^{i\nu(\theta)}$ $(0 \leqslant \theta \leqslant 2\pi, \theta \neq \alpha_j, j = 1, \ldots, n)$,

(ii) $\nu(0) = \alpha$, $\nu(2\pi) - \nu(0) = 2n\pi$,

(iii) $\nu(\theta)$ is right continuous, and left continuous except for jump discontinuities of π at α_j $(j = 1, \ldots, n)$.

Indeed, $\nu(\theta)$ has the form

$$\nu(\theta) = \begin{cases} \alpha + j\pi + n\theta/2 & \text{if } \alpha_j \leqslant \theta < \alpha_{j+1},\ j = 0, 1, \ldots, n, \\ \alpha + 2n\pi & \text{if } \theta = 2\pi, \end{cases}$$

where $\alpha_0 = 0$ and $\alpha_{n+1} = 2\pi$. Then there exists a continuous function ν_0 on $[0, 2\pi]$ such that

(i)′ $\nu_0(\alpha_j) = -\alpha + j\pi - (n/2)\alpha_j$ $(j = 1, \ldots, n)$,

(ii)′ $\nu_0(0) = \nu_0(2\pi) = -\alpha$,

(iii)′ ν_0 is a straight line in each interval $[\alpha_j, \alpha_{j+1}]$ $(j = 0, \ldots, n)$.

Now we can find the desired function k of the lemma. Let ν_0^* be the harmonic conjugate of ν_0; then $\nu_0 + i\nu_0^*$ belongs to A because ν_0 is in a Lipschitz class (cf. [**3**, p. 140]). Let $k = -i\exp(-\nu_0^* + i\nu_0)$; then both k and k^{-1} are in A and

$$\frac{k(e^{i\theta})p(e^{i\theta})}{|k(e^{i\theta})p(e^{i\theta})|} = e^{i(\nu(\theta) + \nu_0(\theta) - \pi/2)}$$

with $-\pi/2 \leqslant \nu(\theta) + \nu_0(\theta) - \pi/2 \leqslant -\pi/2$ $(0 \leqslant \theta \leqslant 2\pi)$.

THEOREM. *If* $p(z) = \prod_{i=1}^{n}(z + a_j)$, $|a_j| \geqslant 1$ $(j = 1, \ldots, n)$, *and* $a_i \neq a_j$ $(i \neq j)$, *then for any nonconstant function* f *in* H^∞ *with* $\|f\|_\infty \leqslant 1$, $p(f)/\|p(f)\|_1$ *is an exposed point of the unit ball of* H^1.

PROOF. Let $\Omega_1 = \{j \mid 1 \leqslant j \leqslant n, |a_j| = 1\}$, $\Omega_2 = \{j \mid 1 \leqslant j \leqslant n, |a_j| > 1\}$, and put $p_i(z) = \prod_{j \in \Omega_i}(z + a_j)$, where $p_i(z) = 1$ if Ω_i is empty $(i = 1, 2)$. By the lemma there exists a k in A such that k^{-1} is in A and $\mathrm{Re}[k(e^{i\theta})p_1(e^{i\theta})] \geqslant 0$ on ∂U. So, $\mathrm{Re}[k(e^{i\theta})p_1(e^{i\theta})] > 0$ on U by the Poisson integral representation of $h(z)p_1(z)$. For any nonconstant f in H^∞ with $\|f\|_\infty \leqslant 1$, $k(f(z))$ is bounded analytic in U, and

$$\mathrm{Re}\left[k(f(z))p_2(f(z))^{-1}p(f(z))\right] = \mathrm{Re}\left[k(f(z))p_1(f(z))\right] > 0$$

on U, and hence $\geqslant 0$ a.e. on ∂U. Then, by Proposition 5(2) of [3], we have that $p(f)/\|p(f)\|_1$ an exposed point of H^1.

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