AN INEQUALITY SUGGESTED BY LITTLEWOOD

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ABSTRACT. It is shown that

$$\sum_n a_n^3 \sum_{m=1}^n a_m^2 \sum_{k=1}^m a_k \leq \frac{3}{2} \sum_n a_n^4 \left[\sum_{k=1}^n a_k \right]^2$$

for arbitrary nonnegative numbers $a_1, a_2, \dots$

The purpose of this note is to prove the following inequality:

$$(1) \quad \sum_n a_n^3 \sum_{m=1}^n a_m^2 A_m \leq K \sum_n a_n^4 A_n^2.$$

Here the a_n 's are arbitrary nonnegative numbers with partial sums $A_n = a_1 + \dots + a_n$ and K is an absolute constant.

The question as to whether any such K exists was offered by Littlewood [3, p.152] as an example of an elementary inequality that is "not easy to decide." Littlewood's paper contains several problems of a similar kind; (1) is perhaps the simplest of these, and certainly the simplest to state. His inequalities all have their roots in the theory of orthogonal series, but they are pursued for their own sake in [3]. The work on orthogonal series, to which Littlewood refers, appears in an unpublished Ph. D. dissertation of Bray [2]. Owing to the technical nature of this analysis, we must refer the reader to [3] for the origins of (1).

Our proof of (1) has no application to orthogonal series, and, in fact, no new techniques are involved. On the other hand, our *search* for a proof of (1) has led to several new inequalities which are of interest in their own right. These inequalities will be described elsewhere [1].

We actually prove a more general result than (1). Setting $p = 2$ and $q = 1$ in the theorem below, and interchanging the order of summation on the left of (3), shows that (1) holds with $K = 3/2$. Furthermore, setting $p = 1$ and $q = 2$ leads to

$$(2) \quad \sum_n a_n A_n^2 \left[\sum_{m \geq n} a_m^{3/2} \right]^2 \leq 4 \sum_n a_n^2 A_n^4,$$

and this settles a second problem of Littlewood [3, p. 152].

I do not know whether the constants appearing in (1), (2), and (3) are best possible; machine calculations indicate that they are not.

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THEOREM. Let $p, q \geq 1$. Then

$$(3) \quad \sum_n a_n^p A_n^q \left[\sum_{m \geq n} a_m^{1+p/q} \right]^q \leq \left[\frac{2p-1}{p} q \right]^q \sum_n [a_n^p A_n^q]^2.$$

PROOF. Our proof involves just two applications of Hölder's inequality; the only point of difficulty lies in the choice of exponents.

We may assume that only finitely many of the a_n 's are positive, say $a_n = 0$ whenever $n > N$. To keep the notation manageable, we set $b_n = a_n A_n^{q/p}$ and $c_n = \sum_{m \geq n} a_m^{1+p/q}$. Elementary estimates give

$$(4) \quad A_n^{1+q/p} \geq b_1 + \cdots + b_n \quad (= B_n \text{ say})$$

and

$$(5) \quad c_n^r - c_{n+1}^r \leq r c_n^{r-1} a_n^{1+p/q},$$

where $r (\geq 1)$ is to be chosen later.

Letting $\theta = (2p-1)^{-1}$ so that $0 < \theta \leq 1$, the left-hand side of (3) may be rewritten as

$$L = \sum_n b_n^{p(1-\theta)} \cdot b_n^{p\theta} c_n^q.$$

Applying Hölder's inequality with exponents $2/(1-\theta)$ and $2/(1+\theta)$, we have

$$\begin{aligned} L &\leq \left[\sum_n b_n^{2p} \right]^{(1-\theta)/2} \left[\sum_n b_n^{2p\theta/(1+\theta)} c_n^{2q/(1+\theta)} \right]^{(1+\theta)/2} \\ &= \left[\sum_n b_n^{2p} \right]^{(1-\theta)/2} \left[\sum_n b_n c_n^r \right]^{(1+\theta)/2}, \end{aligned}$$

where we have set

$$(6) \quad r = 2q/(1+\theta) = (2p-1)q/p \geq 1.$$

Thus, to prove (3), it suffices to show that

$$(7) \quad \sum_n b_n c_n^r \leq r^r \sum_n b_n^{2p}.$$

Summing by parts, and then applying (4) and (5), we see that

$$\begin{aligned} \sum_{n=1}^N b_n c_n^r &= \sum_{n=1}^{N-1} B_n [c_n^r - c_{n+1}^r] + B_N c_N^r \\ &\leq r \sum_{n=1}^N a_n^{1+p/q} A_n^{1+q/p} c_n^{r-1} \\ &= r \sum_{n=1}^N b_n^{1/r} c_n^{r-1} \cdot b_n^{2p/r}, \end{aligned}$$

where $r^* = r/(r - 1)$ is the conjugate of r (see (6)). Applying Hölder's inequality once more gives

$$\sum_n b_n c_n^r \leq r \left[\sum_n b_n c_n^r \right]^{1/r^*} \left[\sum_n b_n^{2p} \right]^{1/r},$$

which is equivalent to (7). $\square$

REFERENCES

1. G. Bennett, *Some elementary inequalities*, Quart. J. Math. (to appear).
2. J. Bray, Ph.D. thesis, Cambridge University.
3. J. E. Littlewood, *Some new inequalities and unsolved problems*, Inequalities (Ed., O. Shisha), Academic Press, New York, 1967, pp. 151–162.

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