REMARKS ON LINEAR DIFFERENTIAL SYSTEMS WITH MEASURABLE COEFFICIENTS

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ABSTRACT. We discuss several topics having to do with linear differential equations with locally integrable coefficients. These include gap labelling and Cantor spectrum for the random Schrodinger operator, and Coppel's perturbation theorem.

1. Introduction. The purpose of this paper is to discuss several matters considered in recent papers dealing with the theory of ordinary linear differential equations

$$(1) \quad x' = y(t)x \quad (x \in \mathbb{C}^n, t \in \mathbb{R})$$

and with the spectral theory of ordinary linear differential operators. The topics we treat are rather diverse, including among others gap labelling for random differential and difference operators, a theorem of Coppel [6] concerning roughness of exponential dichotomy, and Cantor spectrum for the almost periodic Schrodinger operator. Nevertheless, the methods we use provide a unifying theme: we regard y as just one element of a compact, translation invariant set Y of coefficient matrices, and apply techniques of topological dynamics and dynamical systems to the corresponding family of ODEs (1). Thus this paper forms part of a now substantial body of work (e.g., [15, 19, 31, 32]; see [35] for fundamental facts and reference to earlier literature).

Three sections follow this introduction. §2 is devoted to the "random" differential equation (1); i.e., the coefficient $y(t) = y_t$ is a stochastic process with values in the set $\text{gl}(n, \mathbb{C})$ of complex $n \times n$ matrices. We first show how many such equations can be put into a context in which tools of topological dynamics apply. The material is not new and is elementary but it is hoped that its presentation here will be of some use, to someone. In any case, we use it to provide a general framework in which results of several recent papers are valid [7, 15, 16, 19]. These results are often stated under unnecessarily restrictive hypotheses, as we indicate. They include gap labelling; in particular we "explain" a result of Bellissard and Scoppola [3] from our topological point of view. We also discuss the relation between spectrum and exponential dichotomy for linear differential operators [15], and a characterization due to De Concini and Johnson [7] of the algebro-geometric potentials for the Schrodinger [27, 23] and AKNS [1] operators.

In §3 we consider Coppel's perturbation theorem [6]. To state it, recall that (1) is said to have *exponential dichotomy* if there is a projection $P: \mathbb{C}^n \rightarrow \mathbb{C}^n$ and constants $K > 0$, $\alpha > 0$ such that, if $\Phi(t)$ is the fundamental matrix solution of

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(1) with $\Phi(0) = I$, then

$$\|\Phi(t)P\Phi^{-1}(s)\| \leq Ke^{-\alpha(t-s)} \quad (t \geq s),$$

$$\|\Phi(t)(I - P)\Phi^{-1}(s)\| \leq Ke^{\alpha(t-s)} \quad (t \leq s).$$

Coppel shows that, if $y(t)$ is continuous, and if $v(t)$ is continuous with $\|v\|_\infty = \sup_t \|v(t)\| \leq \delta$, then for sufficiently small δ , the perturbed system

$$(1') \quad x' = (y(t) + v(t))x$$

also has exponential dichotomy with, say, projection $P(v)$, and moreover $\|P(v) - P\| \leq K\|v\|_\infty$ where K depends only on equation (1).

Our result is a partial generalization of Coppel's in the sense that $y(t)$ is replaced by a compact, translation-invariant set Y of measurable coefficient matrices, and the perturbation v also lies in such a set. In addition, our proof differs substantially from Coppel's; in particular, we use ideas of Hirsch, Pugh, and Shub [14] and the spectral theorem for linear operators to prove the Lipschitz estimate on the projection $P(v)$. The Lipschitz continuity in v is especially important, since it can often be "bootstrapped" into actual differentiable dependence of P on v (see, e.g., Palmer [29] or [16] for situations in which this is true).

Sacker and Sell [32] have also proven a roughness result which, however, does not seem to lend itself to a Lipschitz continuity statement. On the other hand, the formulation of our theorem is influenced by the Sacker-Sell statement. Moreover, part of our proof relies on arguments used by them in the proof of their theorem.

Finally, in §4, we consider the one-dimensional Schrodinger operator

$$L = -d^2/dt^2 + q(t)$$

with (Bohr) almost periodic potential $q(t)$. It is conjectured that, for generic q , the spectrum Σ of L is a Cantor set, i.e., if L is viewed as a selfadjoint operator on $L^2(\mathbf{R})$, then $\{\lambda \in \mathbf{C} | \lambda - L \text{ is not invertible}\}$ is a closed nowhere dense subset of $\mathbf{R}$.

We prove a result which supports this conjecture. Namely, given some reasonable Banach space B of almost periodic potentials q , we find a residual subset $B_1 \subset B$ such that, if $q \in B_1$, then $L = L_q$ has at least one of the following properties: (i) $\Sigma = \Sigma_q$ is a Cantor set; or (ii) if $I \subset \Sigma$ is a nondegenerate interval, then the Lyapounov exponent (see §4) $\beta = \beta(\lambda)$ of $L_q\phi = \lambda\phi$ is strictly positive on a subset $I_1 \subset I$ of positive Lebesgue measure. Since $\beta(\lambda)$ measures exponential growth of solutions ϕ of $L_q\phi = \lambda\phi$, one expects that (ii) implies (i); i.e., for a generic perturbation of q' of q , $L_{q'}$ has Cantor spectrum. However we have not been able to prove this.

It should be emphasized that our proof uses neither smoothness of q nor Diophantine conditions on rotation numbers. Instead, we use results of Johnson and Moser [18], Moser and Poschl [26], and De Concini and Johnson [7], which are valid independently of such assumptions.

2. Random differential equations. We consider the differential equations

$$(2)_\omega \quad x' = Z_t(\omega)x \quad (x \in \mathbf{C}^n, \omega \in \Omega),$$

where Ω is a measure space with probability measure μ , and for each $t \in \mathbf{R}$, $\Omega \rightarrow \text{gl}(n, \mathbf{C})$ is a μ -measurable function. We will give conditions on the "random process" $\{Z_t | t \in \mathbf{R}\}$ under which methods of topological dynamics may be used to

study equations $(2)_\omega$; in particular, we are interested in conditions assuring that the solutions generate a continuous cocycle (see 2.3).

We make the following assumptions on $\{Z_t | t \in \mathbf{R}\}$.

(i) (*Stochastic continuity*) Given $\varepsilon > 0$, there exists $\delta > 0$ such that, if $|h| < \delta$, then $\mu\{\omega | \|Z_{t+h}(\omega) - Z_t(\omega)\| > \varepsilon\} < \varepsilon$. Here and below $\|\cdot\|$ is some fixed norm on $\text{gl}(n, \mathbf{C})$. It follows [9] that we can modify each Z_t on a set of μ -measure zero in such a way that the map $(t, \omega) \rightarrow Z_t(\omega)$ is measurable with respect to the product of Lebesgue measure m on $\mathbf{R}$ with μ .

(ii) (*Stationarity*) If $t_1, \dots, t_k \in \mathbf{R}$ and $A_1, \dots, A_k$ are Borel subsets of $\text{gl}(n, \mathbf{C})$, then the joint distribution $\mu\{\omega | Z_{t_1}(\omega) \in A_1, \dots, Z_{t_k}(\omega) \in A_k\}$ depends only on the differences $t_2 - t_1, \dots, t_k - t_{k-1}$.

Let us now embed the functions $t \rightarrow Z_t(\omega)$ in a compact subset Y of a function space E in such a way that μ defines a Radon probability measure on Y . There are many ways of doing this; we single out the following, which has proved to be useful.

For $1 \leq p < \infty$, let $E_p = \{\eta: \mathbf{R} \rightarrow \text{gl}(n, \mathbf{C}) | \eta \text{ is measurable and } \|\eta\|_p < \infty\}$, where

$$\|\eta\|_p^p = \sup_t \int_t^{t+1} \|\eta(s)\|^p ds$$

if $1 \leq p < \infty$, and $\|\eta\|_\infty = \sup_t \|\eta(t)\|$. Thus E_p is a vector subspace of $L_{\text{loc}}^p(\mathbf{R}, \text{gl}(n, \mathbf{C}))$. Let $\eta^{ij}(\cdot)$ be the ij th component of the matrix function $\eta(\cdot)$ ($1 \leq i, j \leq n$). Give E_p the weak topology defined as follows. Let q satisfy $1/p + 1/q = 1$, and let $\phi \in L^q(\mathbf{R})$ have compact support; we say that $\eta_r \rightarrow \eta$ iff for each such ϕ ,

$$\int_{-\infty}^{\infty} \eta_r^{ij}(s) \phi(s) ds \rightarrow \int_{-\infty}^{\infty} \eta^{ij}(s) \phi(s) ds.$$

Next suppose that, for μ -a.a. ω , there exists a constant M independent of ω such that

$$(3_p) \quad \sup_t \int_t^{t+1} \|Z_s(\omega)\|^p ds \leq M.$$

If $p = 1$, suppose in addition that

$$(4) \quad \lim_{\varepsilon \rightarrow 0^+} \sup_t \int_t^{t+\varepsilon} \|Z_s(\omega)\| ds = 0$$

for μ -a.a. ω . Replacing Ω by a subset of μ -measure 1 if necessary, we assume that these conditions hold for all $\omega \in \Omega$.

Define $k: \Omega \rightarrow E_p: k(\omega)(t) = Z_t(\omega)$. Let Y be the closure of $k(\Omega) \subset E_p$ in the weak topology. Assume condition $(3)_p$ (and condition (4) if $p = 1$). Then Y is compact and metrizable.

Let us check that $k: \Omega \rightarrow Y$ is Borel measurable; i.e., if $B \subset Y$ is a Borel set, then $k^{-1}(B)$ is μ -measurable. In fact, let $\phi \in L^q(\mathbf{R})$ have compact support. Fix $1 \leq i, j \leq n$, and

$$f_\phi^{ij}(y) = \int_{\mathbf{R}} y^{ij}(s) \phi(s) ds \quad (y \in Y).$$

Then $f_\phi^{ij} \circ k$ is μ -measurable because $Z_t(\omega)$ is jointly measurable. The subalgebra of $C(Y)$ generated by $\{f_\phi^{ij}\}$ separates points in Y , hence generates $C(Y)$. Hence

$f \circ k$ is μ -measurable for all $f \in C(Y)$, and Borel measurability of k follows from metrizability of Y .

Now define a bounded linear functional $\hat{\mu}$ on $C(Y)$ by

$$\hat{\mu}(f) = \int_{\Omega} (f \circ k)(\omega) d\mu(\omega).$$

Then $\hat{\mu}(f) \geq 0$ when $f \geq 0$, and $\hat{\mu}(1) = 1$. Let ν be an associated Radon probability measure. One shows that $\mu(k^{-1}(B)) = \nu(B)$ for each Borel $B \subset Y$. Thus k defines a measure-homomorphism from (Ω, μ) to (Y, ν) (with the Borel field on Y).

In our construction of (Y, ν) , we have used only stochastic continuity and assumptions (3)_p, (4). We now consider stationarity: this of course translates into invariance of ν under the *translation flow* τ on Y . Here $\tau: Y \times \mathbf{R} \rightarrow Y$ is defined by

$$\tau(y, t)(s) = y(t + s) \quad (y \in Y; t, s \in \mathbf{R}).$$

The map τ is jointly continuous. We write $\tau_t(y) = \tau(y, t)$. The measure ν is said to be *invariant* if $\nu(\tau_t(B)) = \nu(B)$ for each Borel $B \subset Y$.

Using stationarity and simple arguments which we omit, one shows

2.1. *Statement.* The measure ν constructed above is invariant.

Ergodicity of $\{Z_t\}$ also carries over to (Y, ν) . We refer to Doob [9, pp. 464 ff.] for the definition and a discussion of this concept. We only note that, applying his constructions to a stochastically continuous and stationary process $\{Z_t\}$, one obtains a one-parameter group of unitary transformations $U_t: L^2(\Omega, \mu) \rightarrow L^2(\Omega, \mu)$. One says that $\{Z_t\}$ is *ergodic* if the only functions f satisfying $U_t f = f$ for all $t \in \mathbf{R}$ are the constant functions. Now, if $g \in L^2(Y, \nu)$ satisfies $g \circ \tau_t = g$ ν -a.e. for each $t \in \mathbf{R}$, then one can show that $f = g \circ k$ satisfies $U_t f = f$ ($t \in \mathbf{R}$). Hence $g = \text{const}$ ν -a.e. Using a standard argument, one has

2.2. *Statement.* If $\{Z_t\}$ is ergodic (thus in particular stationary), then ν is ergodic; i.e., ν is invariant, and, if $B \subset Y$ is a Borel set such that $\nu(B \Delta \tau_t B) = 0$ for each $t \in \mathbf{R}$ (Δ = symmetric difference), then $\nu(B) = 0$ or $\nu(B) = 1$.

Our attitude now is that the random differential equations $(2)_{\omega}$ have been translated into the family of ODEs

$$(1)_y \quad x' = y(t)x \quad (x \in \mathbf{C}^n, y \in Y).$$

We can apply techniques of topological dynamics to study equations $(1)_y$ because of the following easy result, whose proof is omitted.

2.3. *LEMMA.* Let $Y \subset E_p$ be weakly compact and invariant under τ . Let $\Phi_y(t)$ be the fundamental matrix solution of $(1)_y$ satisfying $\Phi_y(0) = I$. Then $\Phi_{\tau_t(y)}(s) \circ \Phi_y(t) = \Phi_y(t+s)$ ($y \in Y; t, s \in \mathbf{R}$), and $\Phi: Y \times \mathbf{R} \rightarrow \text{GL}(n, \mathbf{C})$ is jointly continuous. That is, Φ defines a continuous cocycle on (Y, τ) [10, 11].

Before considering applications of the construction of (Y, ν) , we make a few observations.

2.4. *REMARKS.* (a) Let Y be a subset of E_p ($p > 1$) for which there exists a constant M such that

$$\sup_t \int_t^{t+1} \|y(s)\|^p ds \leq M \quad (y \in Y).$$

Then the distribution topology on Y agrees with the weak topology.

(b) In [21], Kotani considers the one-dimensional Schrodinger equation

$$\left(\frac{-d^2}{dt^2} + y(t)\right)\phi = \lambda\phi$$

with $y \in H = L^2(\mathbf{R}, (1+|t|^3)^{-1}dt)$. If Y is a weakly compact and translation invariant subset of H , then Y defines a weakly compact and translation invariant subset of E_2 . Thus our framework includes Kotani's if we insist on (weak) compactness of the coefficient space Y .

(c) One might prefer to do without condition (4) if $p = 1$. The basic problem if one omits (4) is the following: The weak closure of a translation-invariant set satisfying (3)₁ need not be weakly compact in E_1 . If we enlarge E_1 to include measure-valued maps y , then we obtain a compact Y , but the cocycle $\Phi_y(t)$ need not be continuous in t .

Let us now briefly consider how certain results of [7, 15, 16, 19] can be applied to random ordinary differential equations of type (2)_ω.

2.5. *Spectrum and exponential dichotomy.* Consider the one-dimensional, random Schrodinger equation

$$L_\omega\phi = \left(\frac{-d^2}{dt^2} + Z_t(\omega)\right)\phi = \lambda\phi,$$

or in system form

$$x' = \begin{pmatrix} \phi \\ \phi' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\lambda + Z_t(\omega) & 0 \end{pmatrix} x.$$

Assume that

$$\omega \rightarrow \begin{pmatrix} 0 & 1 \\ -\lambda + Z_t(\omega) & 0 \end{pmatrix}$$

satisfies (3)_p for some $p \geq 1$ (and (4) if $p = 1$) for some (hence all) $\lambda \in \mathbf{C}$. We obtain differential equations

$$(5)_y \quad x' = \begin{pmatrix} 0 & 1 \\ -\lambda + y(t) & 0 \end{pmatrix} x \quad (y \in Y),$$

where Y is a weakly compact, translation-invariant subset of $L^p_{\text{loc}}(\mathbf{R})$.

Using arguments of [15], we now obtain the following results.

(i) If $\text{Im } \lambda \neq 0$, then equations (5)_y have exponential dichotomy. Thus each Schrodinger operator $L_y = -d^2/dt^2 + y(t)$ is in the limit-point case at $t = \pm\infty$ ([5]; this follows from known limit-point criteria), and nonzero solutions $\phi_\pm(t)$ of $L_y\phi = \lambda\phi$ which are in $L^2(0, \pm\infty)$ decay exponentially as $t \rightarrow \pm\infty$ at an exponential rate which is bounded away from zero independent of $y \in Y$.

(ii) If the orbit $\{\tau_t(y) | t \in \mathbf{R}\}$ is dense in Y , then $\{\lambda \in \mathbf{R} | (\lambda - L_y)^{-1} \text{ is bounded in } L^2(\mathbf{R})\} \equiv \text{resolvent of } L_y \text{ equals } \{\lambda \in \mathbf{R} | \text{equations (5)}_y \text{ have ED}\}$. If ν is ergodic and $Y = \text{Support}(\nu)$, then, for ν -a.a. y , the orbit through y is dense in Y . One can show that, in our construction of (Y, ν) , $k^{-1}(\text{Support } \nu)$ has μ -measure 1 in Ω if (Ω, μ) is ergodic. Thus for a stationary ergodic process $\{Z_t\}$ as considered here, almost all operators L_ω satisfy the hypothesis of (ii).

2.6. *Gap labelling.* One can use 2.5 to study "gap labelling," which refers to a systematic assignment of real numbers to intervals in the resolvent of L_ω . Suppose

that μ is ergodic, so that ν is also, and that $Y = \text{Support}(\nu)$. Following Schwarzmann [33], introduce the group $\check{H}^1(Y; \mathbf{Z})$ of equivalence classes of continuous maps $\phi: Y \rightarrow K = \text{unit circle}$ under the equivalence relation $\phi \sim \psi \Leftrightarrow \phi(y)\psi(y) = \exp iR(y)$ for a continuous function $R: Y \rightarrow \mathbf{R}$. The group operation on $\check{H}^1(Y; \mathbf{Z})$ is induced by pointwise multiplication of mappings. Define

$$h: \check{H}^1(Y; \mathbf{Z}) \rightarrow \mathbf{R}: [\phi] \rightarrow \lim_{t \rightarrow \infty} \frac{1}{t} \text{Arg } \phi(\tau_t(y)),$$

where $[\]$ denotes equivalence class and the limit exists for ν -a.a. y [33]. Now, it is shown in [15] that, if $y \in Y$ has dense orbit, then each $\lambda \in \text{Resolvent}(L_y)$ gives rise to a continuous function ϕ_λ from Y to the real projective space $P^1(\mathbf{R}) \cong K$. In fact, ϕ_λ arises in the following way. Begin with the relation $\text{Resolvent}(L_y) = \{\lambda \in \mathbf{C} \mid \text{equations (5)}_y \text{ have ED}\}$. Since the coefficient matrix in (5) $_y$ has trace zero, the projection $P = P_y: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ in the definition of ED must have one-dimensional range. Let l_y be the range of P_y ; then l_y is a line through the origin in $\mathbf{R}^2$, i.e. an element of $P^1(\mathbf{R})$. Define $\phi_\lambda: Y \rightarrow P^1(\mathbf{R}): y \rightarrow l_y$. It turns out that $\alpha(\lambda) =_{\text{def}} h(\phi_\lambda) \in \mathbf{R}$ gives a labelling of the gaps: $\alpha(\lambda)$ depends only on the gap I containing λ , and is 1-1 on the set of all gaps. We conclude that gaps can be labelled by the rotation number [18] $\alpha(\lambda)$ for the random differential equations (2) $_\omega$ when μ is ergodic.

These ideas can be used to "explain" geometrically a gap-labelling result of Bellissard and Scoppola [3], obtained using C^* -algebra techniques. They consider the (difference) operator equation

$$(*) \quad L(x)_n = x_{n+1} + x_{n-1} + v(n)x_n = \lambda x_n \quad (n \in \mathbf{Z}),$$

where $v(n) = \tilde{v}(\theta_0 + n)$ for a measurable but noncontinuous function $\tilde{v}$ on the circle $K = \{\theta \mid 0 \leq \theta < 2\pi\}$. As in [15], one can label the gaps for L by using the Schwarzmann homomorphism on the suspension $\hat{V}$ of the hull V of the function v .

2.7. *Weyl m -functions and algebro-geometric potentials.* Once again consider the Schrodinger equation

$$L_y \phi = \left(\frac{-d^2}{dt^2} + y(t) \right) \phi = \lambda \phi,$$

where $y \in Y \subset E_p$ is weakly compact and translation-invariant. Let ν be an ergodic measure on Y , and suppose that $\text{Support}(\nu) = Y$. Let $\Phi_\pm(t) \in L^2(0, \pm\infty)$ satisfy $L_y \phi = \lambda \phi$ for $\text{Im } \lambda \neq 0$, and define the Weyl m -functions

$$m_\pm(y, \lambda) = \phi'_\pm(0)/\phi_\pm(0) \quad (\text{Im } \lambda \neq 0).$$

There is a relation between the behavior of $m_\pm$ as $\text{Im } \lambda \rightarrow 0$ and the Lyapounov number $\beta = \beta(\lambda)$ of (5) $_y$:

$$\beta(\lambda) = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \|\Phi_y(t)\|,$$

where $\Phi_y(t)$ is the fundamental matrix solution of (5) $_y$ satisfying $\Phi_y(0) = I$ (the limit exists and is independent of y for μ -a.a. y , by the Oseledec theorem). In fact, De Concini and Johnson [7] show that, if $\beta(\lambda) = 0$ for a.a. λ in an interval I , then $\lambda \rightarrow m_+(y, \lambda)$ extends holomorphically from $\{\lambda \mid \text{Im } \lambda > 0\}$ through I , and the extension equals $m_-(y, \lambda)$ for $\text{Im } \lambda < 0$. A similar change of sign occurs if one begins

with $m_-(y, \lambda)$ in the upper half-plane. Using this result, the well-known algebro-geometric potentials [27, 23] $y(t)$ are characterized in a simple and surprising way in the class of all pairs (Y, μ) : namely as those processes for which the spectrum Σ of L_y is a finite union of intervals for μ -a.a. y , and for which $\beta(\lambda) = 0$ a.e. in Σ .

By the construction presented above, we thus obtain an analogous characterization of the algebro-geometric potentials in the class of ergodic random Schrodinger operators L_ω .

From the present discussion and from [7], one also obtains a characterization of the algebro-geometric AKNS potentials (named after [1]) in the class of ergodic random AKNS potentials.

3. Coppel's theorem. Our starting point in this section is a compact metric space Y with continuous flow τ , and a continuous cocycle (see 2.4) $\Phi: Y \times \mathbf{R} \rightarrow \text{GL}(n, \mathbf{C})$. For example, Y might be a weakly compact subset of one of the spaces E_p ($p \geq 1$) of §2, and Φ might be the cocycle defined by the corresponding family $(1)_y$ of differential equations.

Suppose that the cocycle Φ has exponential dichotomy (we abbreviate to ED). The definition is a natural generalization of that given in §1: there are constants $K > 0, \alpha > 0$ and a continuous function $P: y \rightarrow P_y$ with domain Y and range equal to the set of projections on $\mathbf{C}^n$ with the following properties:

- (i) $\|\Phi_y(t)P_y\Phi_y^{-1}(s)\| \leq Ke^{-\alpha(t-s)}$ ($t \geq s; y \in Y$);
- (ii) $\|\Phi_y(t)(I - P_y)\Phi_y^{-1}(s)\| \leq Ke^{\alpha(t-s)}$ ($t \leq s; y \in Y$).

It is often important to know that, if another cocycle $\Psi: Y \times \mathbf{R} \rightarrow \text{GL}(n, \mathbf{C})$ is "close" to Φ , then Ψ also has ED, the corresponding projections are close to those of Φ , and the projections vary with some degree of smoothness in Ψ . Both Coppel's result [6] and ours are of this type. As noted in the Introduction, the formulation of Sacker and Sell's roughness result [32] does not include a Lipschitz continuity statement.

3.1. THEOREM. *Let (Y, t) be a flow with Y compact metric, i.e., $\tau: Y \times \mathbf{R} \rightarrow Y$ is continuous, $\tau(y, 0) = y$, and $\tau_t \circ \tau_s = \tau_{t+s}$ where $\tau_t(y) \equiv \tau(y, t)$ ($y \in Y; t, s \in \mathbf{R}$). Let $\Psi: Y \times \mathbf{R} \rightarrow \text{GL}(n, \mathbf{C})$ be another cocycle, and let*

$$\delta = \sup_y \sup_{-1 \leq t \leq 1} \|\Phi_y(t) - \Psi_y(t)\|.$$

Then there exists $\delta_1 > 0$ such that, if $\delta \leq \delta_1$, then Ψ has ED with, say, family of projections $\{Q_y | y \in Y\}$; moreover, there exists a constant K depending only on Φ such that $\sup_y \|P_y - Q_y\| \leq K\delta$.

The last part of Theorem 3.1 expresses Lipschitz continuity of $Q: y \rightarrow Q_y$ in the cocycle Ψ .

PROOF OF THEOREM 3.1. Our proof is motivated by arguments of Hirsch, Pugh, and Shub [14]. Let B be the Banach space of all bounded (not necessarily continuous) maps $b: Y \rightarrow \mathbf{C}^n$ with norm $\|b\|_\infty = \sup_y |b(y)|$. Here $|\cdot|$ is the Euclidean norm on $\mathbf{C}^n$. Define a one-parameter group of bounded linear transformations

$$T_\Phi(t): B \rightarrow B: T_\Phi(t) \cdot b(y) = \Phi(\tau_{-t}(y), t) \cdot b(\tau_{-t}(y)).$$

Define $\tilde{P}: B \rightarrow B$ by $(\tilde{P}b)(y) = P_y(b(y))$ for $b \in B$. Then $\tilde{P}$ is a continuous projection on B , and it follows from ED that $\tilde{P}$ commutes with $T_\Phi(t)$ for all $t \in \mathbf{R}$.

By ED and the spectral radius formula [30], the spectrum of $T_\Phi(t)$ restricted to $\text{Im } \tilde{P}$ is contained in the open disc with radius $e^{-\alpha t/2}$. Similarly, the spectrum of $T_\Phi(-t) = T_\Phi(t)^{-1}$ restricted to $\text{Ker } \tilde{P}$ is contained in this disc. Therefore the unit circle C separates the spectrum of $T_\Phi(1)$ into two disjoint closed sets, one inside C and one outside C .

Next let C be positively oriented. Define a projection $P_*: B \rightarrow B$ by

$$P_* = \frac{1}{2\pi i} \int_C (\lambda - T_\Phi(1))^{-1} d\lambda.$$

By [30, Chapter 12] and continuity of $\Phi_y(t)$, the image of P_* equals $\{b \in B \mid \lim_{t \rightarrow \infty} \|T_\Phi(t) \cdot b\|_\infty = 0\}$, and the kernel of P_* equals $\{b \in B \mid \lim_{t \rightarrow \infty} \|T_\Phi(-t) \cdot b\|_\infty = 0\}$.

Define $T_\Psi(t)$ to be the one-parameter group of bounded linear transformations on B defined by the cocycle Ψ . We can choose $\delta_1 > 0$ (depending only on Φ) such that, if $\|T_\Psi(1) - T_\Phi(1)\| < \delta_1$, then the spectrum of $T_\Psi(1)$ is also separated into two closed disjoint subsets by C . Let

$$Q_* = \frac{1}{2\pi i} \int_C (\lambda - T_\Psi(1))^{-1} d\lambda,$$

and let $M = \|(\lambda - T_\Phi(1))^{-1}\|_\infty$, where the norm is of course the operator norm induced by $\|\cdot\|_\infty$ on B . Then

$$\|Q_* - P_*\|_\infty \leq M \sum_{n=1}^{\infty} M^n \cdot \|T_\Psi(1) - T_\Phi(1)\|_\infty^n.$$

Choosing, for example, $\delta_1 \leq 1/2M$, and $K = 2M^2$, we obtain with $\delta = \|T_\Phi(1) - T_\Psi(1)\|_\infty$:

$$\|Q_* - P_*\|_\infty \leq K\delta.$$

Now we show that $(Q_*b)(y) = Q_y(b(y))$ where $Q_y: \mathbb{C}^n \rightarrow \mathbb{C}^n$ is a projection ($y \in Y$). To do so, note first that (perhaps shrinking δ_1 if necessary) we can assume that there is a bounded linear map $h: \text{Range } P_* \rightarrow \text{Ker } P_*$ such that $\text{Range } Q_* = \{b + h(b) \mid b \in \text{Range } P_*\}$. In fact, $h(b)$ can be characterized as the unique element of $\text{Ker } P_*$ such that $\|T_\Psi(t)(b + h(b))\|_\infty \rightarrow 0$ as $t \rightarrow +\infty$ [30].

This being said, let $y \in Y, v \in \text{Range } P_y$, and set

$$b_v(\tilde{y}) = \begin{cases} v, & \tilde{y} = y, \\ 0, & \tilde{y} \neq y. \end{cases}$$

Define $h_y: \text{Range } P_y \rightarrow \text{Ker } P_y$ by $h_y(v) = h(b_v)(y)$. Let $b \in \text{Range } P_*$. Using the uniqueness of $h(b)$, the definition of $T_\Psi(t)$, and the characterization $\text{Range } Q_* = \{c \in B \mid \lim_{t \rightarrow \infty} \|T_\Psi(t) \cdot c\|_\infty = 0\}$, we see that $h(b)(y) = h_y(b(y))$ ($y \in Y$).

Shrinking δ_1 again, we can find a bounded linear map $p: \text{Ker } P_* \rightarrow \text{Range } P_*$ such that $\text{Ker } Q_* = \{b + p(b) \mid b \in \text{Ker } P_*\}$. Reasoning much as we did just above, we can write $p(b)(y) = p_y(b(y))$. It now follows that $(Q_*b)(y) = Q_y(b(y))$ where Q_y is a projection ($b \in B, y \in Y$). It follows from this statement and [30, Chapter 12] that (i) and (ii) hold with Ψ_y, Q_y in place of Φ_y, P_y .

It must still be shown that $y \rightarrow Q_y$ is continuous. First of all, write $Y = Y_0 \cup Y_1 \cup \cdots \cup Y_n$ where $Y_i = \{y \in Y \mid \dim \text{range } P_y = i\}$. It follows from continuity of $y \rightarrow P_y$ and (i), (ii) that each Y_i is an open and closed, invariant subset of Y .

Shrinking δ_1 and using $\|Q_* - P_*\|_\infty \leq K\delta_1$, we can assume $\dim Q_y = \dim P_y$ ($y \in Y$). Next note that, if $\sup_t \|\Psi_y(t)x\| < \infty$ for some $y \in Y$ and $x \in \mathbf{C}^n$, then $x = 0$. Since $\dim Q_y$ is constant on each minimal subset of Y_i , a result of Sacker and Sell [32] implies that Ψ has ED on Y_i ($0 \leq i \leq n$). Thus $y \rightarrow Q_y$ is continuous on each Y_i , hence on all of Y .

Let us now show how Theorem 3.1 can be applied to systems like those considered in §2. For $p \geq 1$, let $Y \subset E_p$ be a weakly compact invariant set. Let Φ_y be the cocycle defined by equations (1)_y. Let V be a weakly compact, invariant subset of E_r for $r \geq 1$. Consider the product space $Y \times V$ with the product flow $\hat{\tau}$: thus $\hat{\tau}((y, v), t) = (\tau_t(y), \tau_t(v))$. Define $\eta_1: Y \times V \rightarrow Y: (y, v) \rightarrow y$ and $\eta_2: Y \times V \rightarrow E_{\min(r, p)}: (y, v) \rightarrow y + v$. Then η_1 and η_2 are flow-preserving maps. If $z = (y, v) \in Y \times V$, define $\Phi_z(t) = \Phi(\eta_1(z), t)$ and $\Psi_z(t) = \Gamma(t)$, where $\Gamma(t)$ is the fundamental matrix solution of $x' = (y(t) + v(t))x$ satisfying $\Gamma(0) = I$. Then Φ and Ψ are continuous cocycles on $Y \times V$ with values in $\text{GL}(n, \mathbf{C})$.

Given E_p the norm $\|\cdot\|_p$ defined by $\|w\|_\infty = \sup_t \|w(t)\|$ if $p = \infty$,

$$\|w\|_p^p = \sup_t \int_t^{t+1} \|w(s)\|^p ds \quad \text{for } 1 \leq p < \infty.$$

One checks using Gronwall's inequality that

$$\sup_{-1 \leq t \leq 1} \|\Phi_z(t) - \Psi_z(t)\| \leq M\|v\|_r, \quad z = (v, y),$$

where M is a constant depending only on the cocycle Φ . Thus if $\sup\{\|v\|_r | v \in V\}$ is sufficiently small, then Theorem 3.1 may be applied: if Φ has ED, then so does Ψ , and the projections are Lipschitz in v with respect to the norm $\|\cdot\|_r$.

We finish the section by considering a frequently-encountered situation. Consider

$$(6) \quad x' = A(t)x \quad (x \in \mathbf{C}^n),$$

where A is uniformly locally L^p ($p \geq 1$). Suppose that $Y = \text{cls}\{\tau_t(A) | t \in \mathbf{R}\}$ is compact in E_p (this is automatic if $p > 1$; see §2). Suppose that (3) has ED with projection P . Then so does the set of equations (6)_y, where y runs over Y . Let $B(t)$ be a function which is uniformly locally L^r , and such that $V = \text{cls}\{\tau_t(B) | t \in \mathbf{R}\}$ is compact in E_r . Suppose that $\sup\{\|v\|_r | v \in V\} = \sup\{\|\tau_t(B)\|_r | t \in \mathbf{R}\} < \varepsilon$, where ε is chosen so that M_ε is less than the δ_1 in Theorem 3.1. Then the perturbed equation

$$x' = (A(t) + B(t))x$$

has ED, and the corresponding projection Q satisfies

$$\|Q - P\| \leq K \cdot \sup_t \|\tau_t(B)\|_r$$

where K depends only on equation (6).

4. The almost-periodic Schrodinger equation. Let $q(t)$ be a real-valued, Bohr almost-periodic function of t , and consider the Schrodinger operator $L = -d^2/dt^2 + q(t)$ with potential q . The corresponding Schrodinger equation is then

$$L\phi = \lambda\phi \quad (\lambda \in \mathbf{C}),$$

or in system form

$$x' = \begin{pmatrix} \phi \\ \phi' \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\lambda + q(t) & 0 \end{pmatrix} x.$$

The spectrum Σ of L (as a selfadjoint operator on $L^2(\mathbf{R})$) turns out to be a non-trivial object to study. One expects that, for generic q , Σ is a Cantor-like subset of $\mathbf{R}$ (i.e., closed and nowhere dense). Our purpose here is to prove a partial result in this direction.

To formulate it, let $Y = \text{cls}\{\tau_t(q) | t \in \mathbf{R}\}$ be the hull of q . Unlike the previous sections, the closure here is taken with respect to the topology of uniform convergence on all of $\mathbf{R}$. Then Y is compact, and translation τ defines a continuous flow on Y . For fixed $\lambda \in \mathbf{C}$, we obtain a family of equations

$$(5)_y \quad x' = \begin{pmatrix} 0 & 1 \\ -\lambda + y(t) & 0 \end{pmatrix} x \quad (y \in Y).$$

It is known [10] that Y can be given the structure of a compact, abelian topological group in such a way that q is the identity and $(*)$ denotes group multiplication $(\tau_t q) * (\tau_s q) = \tau_{t+s} q$ ($t, s \in \mathbf{R}$). Let μ be normalized Haar measure on Y .

Now we introduce the Lyapounov exponent $\beta(\lambda)$ of equations $(5)_y$. For fixed $\lambda \in \mathbf{C}$, let $\Phi_y(t)$ be the fundamental matrix solution of $(5)_y$ such that $\Phi_y(0) = I$. Define

$$\beta(\lambda) = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \|\Phi_y(t)\|;$$

the Oseledec theorem [28] implies that the limit exists and is independent of y for μ -a.a. y .

It is known that λ is in the resolvent of $L_y = -d^2/dt^2 + y(t)$ if and only if equations $(5)_y$ have ED [17]. It is also known that, if $\beta(\lambda) > 0$, and if equations $(5)_y$ do not have ED, then the solutions display quite complicated behavior. In fact, it was not known until Millionschikov's example [35] of 1969 that an almost periodic ordinary differential equation without ED whose coefficient matrix has trace zero can have strictly positive Lyapounov exponent (see also Vinograd [36]).

We will show that, for generic q , one of the following holds.

(i) The spectrum Σ of $L = -d^2/dt^2 + q(t)$ is a Cantor set.

(ii) If $I \subset \Sigma$ is an interval, then $\beta(\lambda) > 0$ for λ in a subset of I of positive Lebesgue measure; thus equation $(5)_y$ has the property that $\|\Phi_y(t)\|$ grows exponentially for μ -a.a. y .

We formulate our theorem so as to make clear that smoothness assumptions on q and Diophantine conditions do not play a role. Thus suppose

$$q(t) = Q(\alpha_1 t, \dots, \alpha_s t) \quad (t \in \mathbf{R}),$$

where $Q(\theta_1, \dots, \theta_s)$ is 2π -periodic in each of the variables $\theta_1, \dots, \theta_s$ ($s \geq 2$). Suppose moreover that $\alpha_1, \dots, \alpha_s$ are independent over the rational numbers $\mathbf{Q}$. Let $r > 0$, and introduce the set D_r of multiply periodic functions Q which are analytic and uniformly bounded in the complex strip $|\text{Im } \theta_i| < r$ ($1 \leq i \leq s$), with period 2π in each complex variable $\theta_1, \dots, \theta_s$. Then D_r is a Banach space in the sup-norm $\|\cdot\|_\infty$.

4.1. PROPOSITION. For $Q \in D_r$, write $q(t) = Q(\alpha_1 t, \dots, \alpha_s t)$. There is a residual subset of $Q \in D_r$ with the property that, if $I \subset \mathbf{R}$ is an interval in the spectrum $\Sigma = \Sigma_Q$ of $L = -d^2/dt^2 + q(t)$, then $\beta(\lambda) > 0$ on a subset of I of positive Lebesgue measure.

PROOF. Let $\{I_k | k \geq 1\}$ be an enumeration of the open bounded subintervals of $\mathbf{R}$ with rational endpoints. Let $F_k = \{Q \in D_\tau | \beta(\lambda) \equiv \beta_Q(\lambda) = 0 \text{ for a.a. } \lambda \in I_k\}$.

We first show that F_k is closed in D_τ for each k . To do so, we appeal to the theory of the Floquet exponent [18, 20, 7]. Briefly, there is a holomorphic function $w(\lambda) \equiv w_Q(\lambda)$, defined for $\text{Im } \lambda > 0$, having positive real part there (and also positive imaginary part), such that, for all $\lambda \in \mathbf{R}$, $\lim_{\varepsilon \rightarrow 0^+} \text{Re } w(\lambda + i\varepsilon) = \beta(\lambda)$. The function w is uniformly bounded on each set of the form $\{\lambda | \text{Im } \lambda > 0, |\lambda| < R\}$, and the bound depends only on R and on $\|Q\|_\infty$.

Now let $Q_l \rightarrow Q$ in D_τ , and write $\beta_l = \beta_{Q_l}$, $w_l = w_{Q_l}$. Suppose each Q_l is in F_k . Then each w_l extends holomorphically to the disc J_k with diameter I_k : this is because iw_l can be Schwarz-reflected through I_k . Writing $\tilde{w}_l$ for the extension, we have $\text{Re } \tilde{w}_l(\lambda) = 0$ for $\lambda \in I_k$. Moreover, using the previous paragraph, $\{\tilde{w}_l | l \geq 1\}$ is uniformly bounded on J_k . Since one can show without difficulty that $w_l(\lambda) \rightarrow w_Q(\lambda)$ for $\text{Im } \lambda > 0$ (e.g., [20]), the limit $\lim_{l \rightarrow \infty} \tilde{w}_l(\lambda) \equiv \tilde{w}(\lambda)$ exists for each $\lambda \in J_k$, is holomorphic on J_k , and $\tilde{w}(\lambda) = w_Q(\lambda)$ for $\text{Im } \lambda > 0$. Hence $\beta_Q(\lambda) = 0$ for all $\lambda \in I_k$. It follows that $Q \in F_k$; i.e., F_k is closed, as we wanted to show.

Next we show that F_k is nowhere dense. We use results of Johnson and Moser [18], Moser and Poschl [26], and De Concini and Johnson [7]. Fix $Q \in F_k$, and let Y be the hull of $q(t)$. Then Y may be identified with a quotient of the s -torus defined by the angular variables θ_i , $0 \leq \theta_i \leq 2\pi$, $1 \leq i \leq s$. In fact, to each $y \in Y$ there corresponds $(\theta_1^0, \dots, \theta_s^0)$ such that

$$y(t) = Q(\theta_1^0 + \alpha_1 t, \dots, \theta_s^0 + \alpha_s t).$$

Let $m_\pm(\lambda) = m_\pm(y, \lambda)$ be the Weyl m -functions corresponding to $y \in Y$ ($\text{Im } \lambda \neq 0$). These functions are described in e.g., [5, Chapter 9]; for example, $m_+(\lambda) = \phi'(0)/\phi(0)$ where $0 \neq \phi$ satisfies $(-d^2/dt^2 + y(t))\phi = \lambda\phi$ and $\phi \in L^2(0, \infty)$. Then $m_\pm(\lambda)$ are holomorphic for $\text{Im } \lambda \neq 0$, and $\text{sgn}[\text{Im } m_\pm(\lambda) \cdot \text{Im } \lambda] = \pm 1$. According to [7], the condition $\beta_Q(\lambda) = 0$ a.e. on I_k implies that $\lambda \rightarrow m_+(y, \lambda)$ extends holomorphically from $\{\text{Im } \lambda > 0\}$ through I_k , and the extension $\tilde{m}_+(y, \lambda)$ equals $m_-(y, \lambda)$ for $\text{Im } \lambda < 0$ ($y \in Y$). Moreover the extension is jointly continuous in $Y \times \{\lambda | \text{Im } \lambda \neq 0 \text{ or } \lambda \in I_k\}$, and $\text{Im } \tilde{m}_+(y, \lambda) > 0$ for $\lambda \in I_k$. Analogous statements hold for $\lambda \rightarrow m_-(y, \lambda)$; with the corresponding notation, $\text{Im } \tilde{m}_-(y, \lambda) < 0$ for $\lambda \in I_k$.

Define $\phi_\pm(t) = \exp \int_0^t m_\pm(\tau_s q, \lambda) ds$ for $\lambda \in I_k$. Then the vectors $(\phi_\pm(t), \phi'_\pm(t))^t$ satisfy equation $(5)_q$ (i.e., $y = q$ in equation $(5)_y$). Moreover the complex lines $l_\pm(t)$ determined by these solutions are bounded away from each other on the projective space $\mathbf{P}^1(\mathbf{C})$ for all $t \in \mathbf{R}$ (precisely because $\text{Im } \tilde{m}_+(\tau_s q, \lambda) > 0$ and $\text{Im } \tilde{m}_-(\tau_s q, \lambda) < 0$ for $\lambda \in I_k$ ($s \in \mathbf{R}$)). Still more, if $(\psi(t), \psi'(t))$ is any nonzero *real* solution of $(5)_q$, then the complex line that it determines is bounded away from both $l_+(t)$ and $l_-(t)$ for all $t \in \mathbf{R}$. If $\Phi_q(t)$ is the fundamental matrix solution of $(5)_q$ satisfying $\Phi_q(0) = I$, then $\det \Phi_q(t) \equiv 1$ by Liouville's formula; it follows easily that all solutions of $(5)_q$ are bounded and (if nonzero) bounded away from zero. Hence $\phi_\pm(t)$ are bounded.

Since $s \rightarrow m_\pm(\tau_s q, \lambda)$ are almost periodic, a theorem of H. Bohr (e.g., [12, Chapter 6]) implies that $\phi_\pm(t)$ are almost periodic. Hence all solutions of $(5)_q$ are almost periodic.

It is time to introduce the rotation number $\alpha(\lambda) = -\lim_{\varepsilon \rightarrow 0^+} \operatorname{Im} w(\lambda + i\varepsilon)$ for $\lambda \in \mathbf{R}$ [18, 13]. In [18], it is shown that $-\alpha$ is continuous, monotone nondecreasing, and

$$\alpha(\lambda) = \lim_{t \rightarrow \infty} \frac{1}{t} \operatorname{Arg}(\psi(t) + i\psi'(t))$$

for each real solution $(\psi(t), \psi'(t))^t$ of $(5)_q$. One can show that

$$\alpha(\lambda) = \lim_{t \rightarrow \infty} \frac{1}{t} \operatorname{Arg} \phi_{\pm}(t);$$

see [18] for a proof. Hence

$$(*) \quad \phi_+(t) = |\phi_+(t)| \exp(i\alpha(\lambda)t + \gamma(t)),$$

where $\gamma(t)$ is almost periodic with mean value zero and frequency module [12] contained in the frequency module M of $\phi_+(t)$. These last statements follow from the theorem of Bohr used earlier. We remark that M = frequency module of $q = \{\sum_{i=1}^s n_i \alpha_i | n_i = 0, \pm 1, \dots\}$.

Next, I_k is in the spectrum of L because $\beta = 0$ on I_k (e.g., [20]). By [18], $-\alpha$ increases exactly on the spectrum of L . Thus there exists $\lambda_0 \in I_k$ such that $2\alpha(\lambda_0)$ is in M .

Using this fact and (*), we see that Lemma 6.1 of Moser and Poschl [26] can be applied to $(5)_q$ when $\lambda = \lambda_0$. Thus there is a trigonometric polynomial $P(\theta_1, \dots, \theta_s)$ such that, if $\tilde{q}(t) = Q(\alpha_1 t, \dots, \alpha_s t) + \varepsilon P(\alpha_1 t, \dots, \alpha_s t)$ and ε is small, then the resolvent of $\hat{L} = -d^2/dt^2 + \tilde{q}(t)$ intersects I_k . Thus $Q + \varepsilon P$ is not in F_k ; i.e. F_k has no interior points.

Finally, let $F_0 = \bigcup_{k=1}^{\infty} F_k$. Then $D_r \setminus F_0$ is a residual subset of D_r such that, if $Q \in D_r \setminus F_0$, then either Σ_Q is a Cantor set, or if $I \subset \Sigma_Q$ is an interval, then $\beta_Q(\lambda) > 0$ on a subset of I of positive Lebesgue measure. This completes the proof of 4.1.

4.2. REMARKS. (i) Of course the argument used above applies to many other Banach spaces of almost periodic functions.

(ii) Using [7 and 13], one can prove the analogue of 4.1 for the AKNS [1] operator

$$L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left[\frac{d}{dt} - A(t) \right]$$

for almost periodic matrices $A(t)$ of trace zero.

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