ON THE STRONG LAW OF LARGE NUMBERS IN BANACH SPACES

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ABSTRACT. We study the relationship between the geometry of a real separable Banach space B (as manifested in its cotype, type, or logtype) and necessary or sufficient criteria for the validity of the Strong Law of Large Numbers (SLLN) for independent B -valued random variables, formulated in terms of the validity of a (verifiable) SLLN for real-valued random variables. Our results are the best possible of their kind and may be used in situations where the SLLN's of Hoffman-Jørgensen and Pisier, and Kuelbs and Zinn are inconclusive.

1. Introduction. Consider a sequence $\{X_n\}$ of independent symmetric random variables taking values in the real separable Banach space $(B, \|\cdot\|)$. $\{\varepsilon_n\}$ will denote an independent sequence of i.i.d. Rademacher random variables ($P(\varepsilon_n = \pm 1) = 1/2, n \geq 1$). $\{X_n\}$ is said to satisfy the strong (weak) law of large numbers ($\{X_n\} \in \text{SLLN}$ (WLLN)) if $\|S_n\| = o(n)$ a.s. ($\|S_n\|/n \xrightarrow{\text{Pr}} 0$), where $S_n = \sum_{j=1}^n X_j$. In [4] type p spaces ($1 \leq p \leq 2$) were characterized as those which the condition $\sum_{j=1}^\infty E\|X_j\|^p/j^p < \infty$ implies the SLLN. A SLLN without geometric restrictions on B was proved in [1]. This theorem, though the best possible, is a consequence of the trivial fact that $\{X_n\} \in \text{SLLN}$ if $\{\|X_n\|\} \in \text{SLLN}$ (see [2] for details). Clearly, therefore, moment or norm assumptions will not yield useful SLLN's in general unless additional assumptions are made. Kuelbs and Zinn [5] hypothesize, for example, that $\{X_n\} \in \text{WLLN}$ and prove that $\{X_n\} \in \text{SLLN}$ if $\|X_j\| \leq Cj/LLj$ and $\sum_{n=1}^\infty \exp(-\varepsilon/\Lambda(n, 2)) < \infty$ for each $\varepsilon > 0$ (Theorem 2), or if $\sum_{n=1}^\infty \Lambda(n, p)^r < \infty$ for some $p \in [1, 2]$ and $r \in (0, \infty)$ (Theorem 1) (where $LLn = \max(1, \log(\log n))$, $\Lambda(n, p) = 1/2^{np} \sum_{j \in I(n)} E\|X_j\|^p$, and $I(n) = \{2^n + 1, \dots, 2^{n+1}\}$). For $B = \mathbb{R}$ the conditions in the first result above are also necessary [9].

Our results are expressed in terms of the validity of a SLLN for real random variables. (The latter can be verified for an arbitrary stabilizing sequence using the criteria in [8, 11].) For example, we define *logtype* p spaces ($1 \leq p \leq 2$) as those in which there is a constant C such that

$$E \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^p \leq C(LLn)^{p-1} \sum_{j=1}^n \|x_j\|^p \quad \text{for each } \{x_j\}_{j=1}^n \text{ in } B,$$

and show that these are precisely the spaces in which the condition

$$(LLn)^{p-1}/n^p \sum_{j=1}^n \|X_j\|^p \rightarrow 0 \quad \text{a.s.}$$

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implies the SLLN. We show that our results (i) are the best of their kind, (ii) are conclusive in cases when most of the above-mentioned SLLN's are not, and (iii) can often be reexpressed in terms of the moments of the individual summands. Nonsymmetric random variables can be included in our discussion by a result of Kuelbs and Zinn [5]: $\{X_n\} \in \text{SLLN}$ iff $\{X_n^s\} \in \text{SLLN}$ and $\{X_n\} \in \text{WLLN}$, where $X_n^s(\omega, \omega') = X_n(\omega) - X'_n(\omega')$ is the symmetrized version of X_n , and X'_n is an independent copy of X_n .

2. Necessary conditions.

THEOREM 1. *B is of cotype q ($2 \leq q < \infty$) iff each sequence $\{X_n\}$ of independent symmetric B -valued random variables satisfying the SLLN also satisfies $\sum_{j=1}^n \|X_j\|^q/n^q \rightarrow 0$ a.s.*

PROOF. Let B be of cotype q and suppose $\{X_n\} \in \text{SLLN}$. It follows that $\{\varepsilon_n x_n\} = \{\varepsilon_n X_n(\omega)\} \in \text{SLLN}$ for almost all ω . As in [4], $E\|\sum_{j=1}^n \varepsilon_j x_j\|^q/n^q \rightarrow 0$ which by the cotype q inequality shows that $\sum_{j=1}^n \|x_j\|^q/n^q \rightarrow 0$, as required. Conversely, let $\{x_n\} \subseteq B$ and set $X_n = n\varepsilon_n x_n$ ($n \geq 1$). By assumption,

$$\sum_{j=1}^n j^q \|x_j\|^q/n^q \rightarrow 0 \quad \text{if} \quad \sum_{j=1}^n j\varepsilon_j x_j/n \rightarrow 0 \quad \text{a.s.},$$

and thus (by Kronecker's Lemma) if $\sum_{j=1}^\infty \varepsilon_j x_j$ converges in $L^q(B)$. Define the Banach spaces $(E, \|\cdot\|_E)$, $(F, \|\cdot\|_F)$ by

$$E = \left\{ x \in B^\infty : \sum_{j=1}^\infty \varepsilon_j x_j \text{ converges in } L^q(B) \right\},$$

$$\|x\|_E = E^{1/q} \left\| \sum_{j=1}^\infty \varepsilon_j x_j \right\|^q$$

and

$$F = \left\{ x \in B^\infty : \sum_{j=1}^n j^q \|x_j\|^q/n^q \rightarrow 0 \right\},$$

$$\|x\|_F = \sup_n \left(\sum_{j=1}^n j^q \|x_j\|^q/n^q \right)^{1/q}.$$

Since the inclusion $E \subset F$ has a closed graph, there exists $C < \infty$ such that

$$\sup_n \left(\sum_{j=1}^n j^q \|x_j\|^q/n^q \right)^{1/q} \leq C \left(E \left\| \sum_{j=1}^\infty \varepsilon_j x_j \right\|^q \right)^{1/q}.$$

Thus for any $\{x_j\}_{j=1}^n$ in B (C is a generic constant),

$$(1) \quad \sum_{j=1}^n j^q \|x_j\|^q/n^q \leq CE \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^q.$$

Fix $N \geq 1$ and let $y_j = 0$ ($1 \leq j \leq N$) and $y_j = x_{j-N}$ ($N < j \leq N+n$), as in [4]. By (1),

$$\sum_{j=1}^n (N+j)^q \|x_j\|^q \leq C(N+n)^q E \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^q.$$

Choosing $N = n$ we see that

$$(1/2)^q \sum_{j=1}^n \|x_j\|^q \leq CE \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^q,$$

proving that B is of cotype q . $\square$

REMARK. A modification of the proof of Lemma 2.3 in [5] shows that if $\|X_j\| \leq j$ ($j \geq 1$) and $\{X_n\} \in \text{WLLN}$, then $E\|S_n/n\|^q \rightarrow 0$ for each $q \geq 1$. Thus the necessary condition in Theorem 1 may be replaced by $\sum_{j=1}^n E\|X_j\|^q/n^q \rightarrow 0$ if $\|X_j\| \leq j$.

EXAMPLE. Let ϕ_n be any sequence increasing to $+\infty$ and define

$$X_j = \varepsilon_j j / (\phi_j)^{1/2q} \quad (j = 2^n, n = 1, 2, \dots)$$

and $X_j = 0$ otherwise. Then $\{X_n\} \in \text{SLLN}$ since $(S_{2^{n+1}} - S_{2^n})/2^n \rightarrow 0$ a.s. [10, p. 159], but $\phi_{2^n}/2^{nq} \sum_{j \in I(n)} |X_j|^q \rightarrow \infty$ a.s. Thus the rate in Theorem 1 cannot be improved.

3. Sufficient conditions. Our results are again in terms of the validity of a real-valued SLLN. We do not place any restrictions on the magnitudes of the norms of the X_n 's as in [5] or assume that $\{X_n\} \in \text{WLLN}$. The latter hypothesis is needed in [5] even in the nonsymmetric case. To see this, consider for $p \in [1, 2)$ the l^p valued sequence $\{X_n\}_{n \geq 1}$ defined by

$$X_n(\omega, k) = \varepsilon_n(\omega)(n/(LLn)^2)^{1/2} I_{\{n\}}(k).$$

$\{X_n\}$ satisfies the hypotheses of Theorem 2 in [5], but $\{X_n\} \notin \text{SLLN}$, since for any ω ,

$$\left\| \sum_{j=1}^n X_j/n \right\| = n^{-1} \left[\sum_{j=1}^n (j/(LLj)^2)^{p/2} \right]^{1/p} \geq Cn^{-1} [n^{p/2+1}/(LLn)^p]^{1/p} \rightarrow \infty.$$

The next lemma is crucial; it is implicit in [6].

LEMMA 1. *Let $\{x_n\}$ be any sequence in a separable Banach space B . Set $S_n = \sum_{j=1}^n \varepsilon_j x_j$ and $s_n^2 = E\|S_n\|^2$. Then there exists $M = M(B)$ such that for each $\varepsilon > 0$,*

$$(2) \quad P(\|S_n\|/s_n > \varepsilon) \leq 3 \exp(-\varepsilon^2/M^2).$$

PROOF. We shall use a basic result of Kwapien [6]: the a.s. convergence of $\sum_{j=1}^\infty \varepsilon_j x_j$ implies that $E(\exp \alpha \|\sum_{j=1}^\infty \varepsilon_j x_j\|^2) < \infty$ for each $\alpha > 0$. Define the

Banach spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ by

$$E = \left\{ x \in B^\infty : \sum_{j=1}^{\infty} \varepsilon_j x_j \text{ converges in } L^2(B) \right\},$$

$$\|x\|_E = \left(E \left\| \sum_{j=1}^{\infty} \varepsilon_j x_j \right\|^2 \right)^{1/2} \quad \text{and}$$

$$F = \left\{ x \in B^\infty : E \exp \left(\alpha \left\| \sum_{j=1}^{\infty} \varepsilon_j x_j \right\|^2 \right) < \infty \text{ for each } \alpha > 0 \right\},$$

$$\|x\|_F = \inf \left\{ t > 0 : E \exp \left(\left\| \sum_{j=1}^{\infty} \varepsilon_j x_j \right\|^2 / t^2 \right) \right\} \leq e.$$

Kwapień's theorem asserts that $E \subset F$. By the closed graph theorem there exists $M < \infty$ such that

$$\inf \{ t > 0 : E \exp(\|S\|^2/t^2) \leq e \} \leq M(E\|S\|^2)^{1/2},$$

where $S = \sum_{j=1}^{\infty} \varepsilon_j x_j$. In other words, for each $\delta > 0$, $E \exp(\|S\|^2/M^2 E\|S\|^2 + \delta) \leq e$, so on letting $\delta \rightarrow 0$,

$$P(\|S_n\|/s_n > \varepsilon) \leq \exp(-\varepsilon^2/M^2) E \exp(\|S_n\|^2/M^2 s_n^2) \leq 3 \exp(-\varepsilon^2/M^2),$$

as required. $\square$

THEOREM 2. *B is of logtype p ($1 \leq p \leq 2$) if and only if each sequence $\{X_n\}$ of independent symmetric B valued random variables satisfying*

$$(LLn)^{p-1}/n^p \sum_{j=1}^n \|X_j\|^p \rightarrow 0 \quad \text{a.s.}$$

also satisfies the SLLN.

PROOF. Assume that $(LLn)^{p-1}/n^p \sum_{j=1}^n \|X_j\|^p \rightarrow 0$ a.s. and let $X_n = Y_n + Z_n$, where $Y_n = X_n I(\|X_n\| \leq j/LLj)$. Then $\{Y_n\}$ and $\{Z_n\}$ are both independent symmetric sequences. As before, $\{Z_n\} \in \text{SLLN}$ iff $\sum_{j \in I(n)} Z_j/2^n \rightarrow 0$ a.s. But

$$\left\| \sum_{j \in I(n)} Z_j/2^n \right\| \leq 1/2^n \sum_{j \in I(n)} I^{p-1}(\|X_j\| > j/LLj) \|X_j\|$$

$$\leq C(LL2^n)^{p-1}/2^{np} \sum_{j \in I(n)} \|X_j\|^p \rightarrow 0 \quad \text{a.s. by assumption.}$$

Next consider the sequence $\{Y_n\}$. Clearly $\{Y_n\} \in \text{SLLN}$ iff $\{\varepsilon_n Y_n(\omega)\} \in \text{SLLN}$ for almost all ω . Choose an ω for which $(LLn)^{p-1}/n^p \sum_{j=1}^n \|x_j\|^p \rightarrow 0$, where $x_n = X_n(\omega)$. Let $y_n = Y_n(\omega)$. We shall show that the sequence $\{\varepsilon_n y_n\}$ satisfies

the hypotheses of Theorem 2 in Kuelbs and Zinn [5] and thus satisfies the SLLN. It is evident that $\|\varepsilon_n y_n\| \leq n/LLn$ ($n \geq 1$). Also,

$$\begin{aligned} LL2^n/4^n \sum_{j \in I(n)} \|y_j\|^2 &= LL2^n/2^{np} \sum_{j \in I(n)} \|y_j\|^p \|y_j\|^{2-p}/2^{n(2-p)} \\ &\leq CLL2^n/2^{np} \cdot 1/(LL2^n)^{2-p} \sum_{j \in I(n)} \|y_j\|^p \\ &= C(LL2^n)^{p-1}/2^{np} \sum_{j \in I(n)} \|y_j\|^p \\ &\leq C(LL2^n)^{p-1}/2^{np} \sum_{j \in I(n)} \|x_j\|^p \rightarrow 0. \end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty} \exp(-\varepsilon 4^n) / \left(\sum_{j \in I(n)} \|y_j\|^2 \right) \leq C_\varepsilon \sum_{n=1}^{\infty} n^{-2} < \infty,$$

so that $\sum_{n=1}^{\infty} \exp(-\varepsilon/\Lambda(n)) < \infty$ for each $\varepsilon > 0$. We need to show that $\{\varepsilon_n y_n\} \in \text{WLLN}$. Since B is of logtype p ,

$$\begin{aligned} P \left(\left\| \sum_{j=1}^n \varepsilon_j x_j \right\| > n\varepsilon \right) &\leq (n\varepsilon)^{-p} E \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^p \\ &\leq C(n\varepsilon)^{-p} (LLn)^{p-1} \sum_{j=1}^n \|x_j\|^p \rightarrow 0. \end{aligned}$$

Thus $\{\varepsilon_n x_n\} \in \text{WLLN}$. Since $(LLn)^{p-1}/n^p \sum_{j=1}^n \|x_j\|^p \rightarrow 0$,

$$\|x_n\| \leq n/(LLn)^{(p-1)/p}$$

if n is large enough. By the remark in §2, $1/nE\|\sum_{j=1}^n \varepsilon_j x_j\| \rightarrow 0$. The comparison principle (Lemma 4.1 in [3]) now shows that $1/nE\|\sum_{j=1}^n \varepsilon_j y_j\| \rightarrow 0$. Thus $\{\varepsilon_n y_n\} \in \text{WLLN}$.

Conversely, consider the Rademacher sequence $\{\varepsilon_n x_n\}, \{x_n\} \subset B$. By hypothesis, $\{\varepsilon_n x_n\} \in \text{SLLN}$ if $(LLn)^{p-1}/n^p \sum_{j=1}^n \|x_j\|^p \rightarrow 0$. Kronecker's lemma and an argument similar to the one in Theorem 1 now show that

$$1/n^p E \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^p \rightarrow 0 \quad \text{if} \quad \sum_{j=1}^{\infty} (LLj)^{p-1} \|x_j\|^p / j^p < \infty.$$

Define the Banach spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ by

$$\begin{aligned} E &= \left\{ x \in B^\infty : \sum_{j=1}^{\infty} (LLj)^{p-1} \|x_j\|^p / j^p < \infty \right\}, \\ \|x\|_E &= \left[\sum_{j=1}^{\infty} (LLj)^{p-1} \|x_j\|^p / j^p \right]^{1/p} \end{aligned}$$

and

$$F = \left\{ x \in B^\infty : 1/n^p E \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^p \rightarrow 0 \right\},$$

$$\|x\|_F = \sup_n 1/n \left(E \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^p \right)^{1/p}.$$

Since the map $E \subset F$ has a closed graph, there exists $C < \infty$ such that

$$\sup_n 1/n \left(E \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^p \right)^{1/p} \leq C \left(\sum_{j=1}^{\infty} (LLj)^{p-1} \|x_j\|^p / j^p \right)^{1/p}.$$

Hence

$$(3) \quad E \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^p \leq Cn^p \left(\sum_{j=1}^n (LLj)^{p-1} \|x_j\|^p / j^p \right) \quad (n \geq 1).$$

As in Theorem 1, fix $N \geq 1$ and let $y_j = 0$ ($1 \leq j \leq N$) and $y_j = x_{j-N}$ ($N < j \leq N+n$). By (3),

$$E \left\| \sum_{j=1}^n \varepsilon_j x_j \right\|^p \leq C(N+n)^p \left(\sum_{j=1}^n [LL(N+j)]^{p-1} \|x_j\|^p / (N+j)^p \right)$$

$$\leq C\{(N+n)^p [LL(N+n)]^{p-1} / (N+1)^p\} \sum_{j=1}^n \|x_j\|^p.$$

The fact that B is of logtype p follows on setting $N = n$. $\square$

EXAMPLES. Let $X_j = \varepsilon_j j / (LLj)^{3/4}$ ($j = 2^n$, $n = 1, 2, \dots$) and $X_j = 0$ otherwise. Then

$$\sum_{j=1}^{\infty} E|X_j|^p / j^p = \infty \quad \text{for each } p \in [1, 2]$$

so that the Hoffmann-Jørgensen and Pisier SLLN [4] is inconclusive. It is also easy to see that $\sum_{n=1}^{\infty} \Lambda(n, p)^r = \infty$ for each $p \in [1, 2]$ and $r \in (0, \infty)$ ($\Lambda(n, p) = 1/2^{np} \sum_{j \in I(n)} E\|X_j\|^p$) so that Theorem 1 in Kuelbs and Zinn [5] is inconclusive as well. On the other hand, for any $p \in [1, 2]$ and $n \in I(k)$,

$$(LLn)^{p-1} / n^p \sum_{j=1}^n |X_j|^p \leq (LL2^{k+1})^{p-1} / 2^{kp} \sum_{j=1}^{2^{k+1}} |X_j|^p$$

$$= (LL2^{k+1})^{p-1} / 2^{kp} \sum_{j=1}^{k+1} 2^{jp} / (LL2^j)^{3p/4}$$

$$\leq C(LL2^k)^{p/4-1} \rightarrow 0 \quad \text{a.s.}$$

so that $\{X_n\} \in \text{SLLN}$.

Our second example shows that the rate in Theorem 2 is the best possible in the sense that for each $p \in [1, 2]$, and for each sequence $\phi(n) \uparrow \infty$, there exists

a sequence $\{X_n\}$ of independent symmetric real random variables satisfying the condition $[(LLn)^{p-1}/n^p \phi(n)] \sum_{j=1}^n |X_j|^p \rightarrow 0$ a.s. but failing the SLLN. Such a sequence may be defined by $X_j = \varepsilon_j 2^n / LL2^n$, $j = 2^n + 1, \dots, 2^n + [LL2^n]$, $n = 1, 2, \dots$, and $X_j = 0$ otherwise. Then for any $k \in I(n)$ and $p \in [1, 2]$,

$$\begin{aligned} & [(LLk)^{p-1}/k^p \phi(k)] \sum_{j=1}^k |X_j|^p \\ & \leq [(LL2^{n+1})^{p-1}/2^{np} \phi(2^n)] \sum_{j=1}^{n+1} [LL2^j] 2^{jp} / (LL2^j)^p \\ & \leq C(LL2^{n+1})^{p-1} / 2^{np} \cdot 2^{(n+1)p} / (LL2^{n+1})^{p-1} \cdot 1/\phi(2^n) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

On the other hand, we have $\Lambda(n) = 1/4^n \cdot [LL2^n] \cdot 4^n / (LL2^n)^2 \geq C/\log n$ so that $\sum_{n=1}^{\infty} \exp(-\varepsilon/\Lambda(n)) = \infty$ for ε small enough. Hence $\{X_n\} \notin \text{SLLN}$, by Prokhorov's result [9].

COROLLARY. *A sequence of independent symmetric random variables with values in a Hilbert space (or more generally a cotype 2, logtype 2 space) satisfies the SLLN if $LLn/n^2 \sum_{j=1}^n \|X_j\|^2 \rightarrow 0$ a.s. and only if $1/n^2 \sum_{j=1}^n \|X_j\|^2 \rightarrow 0$ a.s. Furthermore, the condition $\phi(n)/n^2 \sum_{j=1}^n \|X_j\|^2 \rightarrow 0$ a.s. is neither necessary nor sufficient for the SLLN for any sequence $\phi(n) \uparrow \infty$, $\phi(n) = o(LLn)$. $\square$*

We now ask when our conditions for the SLLN may be rephrased in terms of individual moments.

PROPOSITION. *Assume that $\|X_n\| \leq Cn/LLn$ a.s. ($n \geq 1$) and $p \in [1, 2]$. Then*

$$\begin{aligned} (4) \quad & (LLn)^{p-1}/n^p \sum_{j=1}^n \|X_j\|^p \rightarrow 0 \quad \text{a.s. iff} \\ & [(LLn)^{p-1}/n^p] \sum_{j=1}^n \|X_j\|^p \rightarrow 0 \quad \text{in probability} \end{aligned}$$

and

$$(5) \quad \sum_{n=1}^{\infty} \exp \left[-\varepsilon 2^{2np} / \left\{ (LL2^n)^{2p-2} \sum_{j \in I(n)} E(\|X_j\|^p - \|X'_j\|^p)^2 \right\} \right] < \infty \quad (\varepsilon > 0),$$

where X'_j is an independent copy of X_j ($j \geq 1$). (Here, (4) may be expressed in terms of the individual moments by the classical degenerate convergence criterion.)

PROOF. As mentioned in §1, $(LLn)^{p-1}/n^p \sum_{j=1}^n \|X_j\|^p \rightarrow 0$ a.s. if and only if (4) holds and

$$(6) \quad (LLn)^{p-1}/n^p \sum_{j=1}^n (\|X_j\|^p - \|X'_j\|^p) \rightarrow 0 \quad \text{a.s.}$$

We have

$$\|\|X_j\|^p - \|X'_j\|^p\| \leq \frac{2Cj^p}{(LLj)^p} \leq \frac{C'j^p/(LLj)^{p-1}}{LL(j^p/(LLj)^{p-1})},$$

so that we may apply a result of Loève [7, p. 258] to conclude that (5) and (6) are equivalent. The assertion follows.

Notice that $E(\|X_j\|^p - \|X'_j\|^p)^2 = 2 \operatorname{Var}(\|X_j\|^p) \leq 2E\|X_j\|^{2p}$, so that

$$\begin{aligned} & (LL2^n)^{2p-2} \sum_{j \in I(n)} E(\|X_j\|^p - \|X'_j\|^p)^2 / 2^{2np} \\ & \leq 2(LL2^n)^{2p-2} \sum_{j \in I(n)} E\|X_j\|^{2p} / 2^{2np} \\ & \leq C(LL2^n)^{2p-2} \sum_{j \in I(n)} E\|X_j\|^2 (2^n / LL2^n)^{2p-2} / 2^{2np} = C\Lambda(n, 2) \end{aligned}$$

so that (5) is a slight improvement of the Kuelbs-Zinn condition. On the other hand, (4) may do worse than their WLLN hypothesis.

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