THE PRESERVATION OF ATRIODICITY BY SEMICONFLUENT MAPPINGS

JAMES F. DAVIS

ABSTRACT. Grace and Vought have proven that the image of a chainable continuum under a semiconfluent mapping is atriodic. This motivates the question of whether the semiconfluent image of an atriodic continuum is atriodic. In this paper this question is answered affirmatively.

In [1] Grace and Vought prove that the semiconfluent image of an arc-like continuum is atriodic. Since arc-like continua are atriodic, the question of whether semiconfluent mappings preserve atriodic continua naturally arises (Problem 8.5 of [5]). The main result of this paper is an affirmative answer to this question. Grace and Vought give an example [1, Example 1, p. 157] of a weakly confluent mapping from an arc-like continuum onto a simple triod. This shows that the result of this paper cannot be extended to weakly confluent mappings.

Definitions and statement of principal result. All spaces considered in this paper are metric. A *continuum* is a compact connected space and a *mapping* is a continuous function. A mapping f of a continuum X onto a continuum Y is *confluent* if for each subcontinuum K of Y , each component of $f^{-1}(K)$ is mapped by f onto K ; if for each subcontinuum K of Y and each two components C_1 and C_2 of $f^{-1}(K)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$, then f is *semiconfluent*; if for each subcontinuum K of Y there is a component of $f^{-1}(K)$ which is mapped by f onto K , then f is *weakly confluent*. Clearly confluent mappings are semiconfluent. Maćkowiak has shown [4, Corollary 3.2, p. 254; 5, Theorem 3.8] that semiconfluent mappings are weakly confluent. A continuum is a *triod* provided that it contains a subcontinuum whose complement has at least three components. A continuum is *atriodic* if it does not contain a triod. If $\mathcal{C}$ is a collection of sets, $\mathcal{C}^*$ denotes the union of the members of $\mathcal{C}$.

The following theorem is our principal result. The initial part of the proof follows very closely the proof given by Grace and Vought for [1, Theorem 1, p. 152].

THEOREM 1. *If f is a semiconfluent mapping from an atriodic continuum X onto the continuum Y , then Y is atriodic.*

PROOF. Suppose that Y contains a triod $W = A \cup B \cup C$, where A, B , and C are continua and $A \cap B = A \cap C = B \cap C = Q$ is a proper subcontinuum of each of A, B , and C . Let a, b , and c be points in $A \setminus Q, B \setminus Q$, and $C \setminus Q$, respectively.

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If there is a component of $f^{-1}(A \cup B)$ which intersects $f^{-1}(a)$ but not $f^{-1}(b)$, then, by the semiconfluence of f , every component of $f^{-1}(A \cup B)$ that intersects $f^{-1}(b)$ also intersects $f^{-1}(a)$. Thus either every component of $f^{-1}(A \cup B)$ that intersects $f^{-1}(a)$ also intersects $f^{-1}(b)$ or every component that intersects $f^{-1}(b)$ also intersects $f^{-1}(a)$. Relabeling if necessary, we will suppose the former. Since f is semiconfluent, it is weakly confluent. Let X_{ac} be a component of $f^{-1}(A \cup C)$ such that $f(X_{ac}) = A \cup C$, and let a' and c' be points in X_{ac} such that $f(a') = a$ and $f(c') = c$. Let X_{ab} be the component of $f^{-1}(A \cup B)$ which contains a' . Thus, X_{ab} contains a point b' such that $f(b') = b$. Let U_a, U_b , and U_c be open subsets of Y containing a, b , and c respectively such that $\overline{U}_a$ and $B \cup C \cup \overline{U}_b \cup \overline{U}_c$, $\overline{U}_b$ and $A \cup C \cup \overline{U}_a \cup \overline{U}_c$, and $\overline{U}_c$ and $A \cup B \cup \overline{U}_a \cup \overline{U}_b$ are mutually exclusive. Let A_0, B_0 , and C_0 be the components of $A \setminus U_a, B \setminus U_b$, and $C \setminus U_c$, respectively, containing Q , and let $W_0 = A_0 \cup B_0 \cup C_0$. Let $V_a = f^{-1}(U_a)$, $V_b = f^{-1}(U_b)$, and $V_c = f^{-1}(U_c)$. Let

$$\mathcal{B} = \{J: J \text{ is a component of } f^{-1}(W_0), J \cap \overline{V}_a \neq \emptyset, \text{ and } J \cap \overline{V}_b \neq \emptyset\},$$

and

$$\mathcal{C} = \{J: J \text{ is a component of } f^{-1}(W_0), J \cap \overline{V}_a \neq \emptyset, \text{ and } J \cap \overline{V}_c \neq \emptyset\}.$$

Since X_{ab} is a continuum intersecting both $f^{-1}(a) \subset V_a$ and $f^{-1}(b) \subset V_b$, $\mathcal{B}$ is nonempty, and since X_{ac} is a continuum intersecting both $f^{-1}(a) \subset V_a$ and $f^{-1}(c) \subset V_c$, $\mathcal{C}$ is nonempty. Suppose that there is a member J of $\mathcal{B}$ such that $f(J)$ and $\overline{U}_c$ are mutually exclusive. From the semiconfluence of f it follows that, if L is a member of $\mathcal{C}$, then $f(L)$ intersects $\overline{U}_b$. Thus either

- (1) if J is in $\mathcal{C}$, $f(J)$ intersects $\overline{U}_b$, or
- (2) if J is in $\mathcal{B}$, $f(J)$ intersects $\overline{U}_c$.

We will suppose (1). The argument in case (2) holds and is similar. Let

$$\begin{aligned} \mathcal{C}_0 = \{N: N \text{ is a component of } f^{-1}(A_0 \cup C_0), \\ N \cap \overline{V}_a \neq \emptyset, N \cap \overline{V}_c \neq \emptyset, \text{ and } N \subset X_{ac}\}. \end{aligned}$$

The remainder of the proof falls into three cases depending on the cardinality of $\mathcal{C}_0$. We will now give an overview of these cases, postponing the details until after some helpful results have been proven. In each case we will reach a contradiction by constructing a triod in X . If $\mathcal{C}_0$ has just one member, then a triod will be constructed with core X_{ac} and arms extend toward $f^{-1}(a), f^{-1}(b)$, and $f^{-1}(c)$. If $\mathcal{C}_0$ has exactly two members, a triod will be constructed with core including X_{ac} , with an arm extending toward $f^{-1}(b)$, and either two arms extending toward $f^{-1}(a)$ or two arms extending toward $f^{-1}(b)$. If $\mathcal{C}_0$ has at least three components, a triod will be constructed with X_{ac} as its core and with three arms extended toward $f^{-1}(b)$.

Preliminaries for the three cases. The following lemmas and theorem will be used in the consideration of the three cases mentioned above. We state them in somewhat more generality than what we need. In particular, note that the mapping in Lemma 2 need not be semiconfluent and that the spaces mentioned in Lemma 2 and Theorem 2 need not be continua.

Through the end of Lemma 2 we adopt the following hypothesis: *Suppose that A and C are subcontinua of the continuum M , $M = A \cup C$, and $Q = A \cap C$ is*

connected. Suppose that a is in $A \setminus C$, c is in $C \setminus A$, and that U_c is an open subset of M which contains c and whose closure does not intersect A . Let C_0 be the component of $C \setminus U_c$ which includes $A \cap C$.

LEMMA 1. *The component of $M \setminus U_c$ which contains a is $A \cup C_0$.*

PROOF. Let K denote the component of $M \setminus U_c$ which contains a . Since $A \cup C_0$ is a connected set which contains a , and $A \cup C_0 \subset (A \cup C) \setminus U_0$, it follows that $A \cup C_0 \subset K$. Now suppose that x is in K . If x is in A then x is in $A \cup C_0$, so suppose that x is not in A . Let C_1 be the closure of the component of $K \setminus A$ which contains x . By [6, Theorem 50, p. 18] C_1 intersects A . Since $C_1 \cap A \subset A \cap C \subset C_0$, it follows that $C_1 \cup C_0$ is connected. Since C_0 is a component of $C \setminus U_c$ and $C_1 \cup C_0 \subset C \setminus U_c$, we have that $C_1 \subset C_0$. Thus x is in C_0 . Hence $K \subset A \cup C_0$, and we have that $K = A \cup C_0$.

LEMMA 2. *Suppose that X is a space, f is a mapping of X onto M , X_{ac} is a component of X , and that a' and c' are points in X_{ac} such that $f(a') = a$ and $f(c') = c$. Let $V_a = f^{-1}(U_a)$ and $V_c = f^{-1}(U_c)$. Then the component of $f^{-1}(A \cup C_0)$ which contains a' is the component of $X_{ac} \setminus V_c$ which contains a' . Moreover, if X_{ac} is compact, this component includes a component of $X_{ac} \setminus (V_a \cup V_c)$ which intersects both $\overline{V}_a$ and $\overline{V}_b$.*

PROOF. Let H_1 denote the component of $f^{-1}(A \cup C_0)$ which contains a' , and let H_2 denote the component of $X_{ac} \setminus V_c$ which contains a' . Since $A \cup C_0 \subset (A \cup C) \setminus U_c$ we have that $f^{-1}(A \cup C_0) \subset f^{-1}(M \setminus U_c) = X \setminus V_c$. Thus $H_1 \subset X \setminus V_c$ and hence $H_1 \subset X_{ac} \setminus V_c$. Therefore $H_1 \subset H_2$. Since $H_2 \subset X \setminus V_c$, it follows that $f(H_2) \subset M \setminus U_c$. The component of $M \setminus U_c$ which contains a is $A \cup C_0$, by Lemma 1, and $f(H_2)$ is a connected subset of $M \setminus U_c$ which contains a . Thus $f(H_2) \subset A \cup C_0$. Therefore $H_2 \subset f^{-1}(A \cup C_0)$. Since H_2 is connected, $H_1 \subset H_2$, and H_1 is a component of $f^{-1}(A \cup C_0)$, we have that $H_1 = H_2$, the first conclusion of the lemma. Now suppose that X_{ac} is compact, that is, it is a continuum. Since X_{ac} intersects V_c , there is a point c'' in $H_1 \cap \overline{V}_c$ by [6, Theorem 52(a), p. 21]. Let T be the component of $H \setminus V_a$ which contains c'' . Again by [6, Theorem 52(a), p. 21], T intersects $\overline{V}_a$ and thus is a component of $X_{ac} \setminus (V_a \cup V_c)$ which intersects both $\overline{V}_a$ and $\overline{V}_c$.

The following separation theorem will be used in Case II of our proof of Theorem 1.

THEOREM 2. *Suppose that X is a compact space, U and V are nonempty open subsets of X whose closures are mutually exclusive, and there are just two components, T_1 and T_2 , of $X \setminus (U \cup V)$ which intersect both $\overline{U}$ and $\overline{V}$. If each of $X \setminus U$ and $X \setminus V$ is the union of two mutually separated sets, one containing T_1 and the other containing T_2 , then X is the union of two mutually separated sets, one containing T_1 and the other containing T_2 .*

PROOF. Let H_1 and H_2 be mutually exclusive closed subsets of X such that $T_1 \subset H_1$, $T_2 \subset H_2$, and $X \setminus V = H_1 \cup H_2$. Let K_1 and K_2 be mutually exclusive

closed subsets of X such that $T_1 \subset K_1$, $T_2 \subset K_2$, and $X \setminus U = K_1 \cup K_2$. Let

$$\begin{aligned}\mathcal{H}_1 &= \{I: I \text{ is a component of } H_1 \text{ and } I \cap \bar{U} \neq \emptyset\}, \\ \mathcal{H}_2 &= \{I: I \text{ is a component of } H_2 \text{ and } I \cap \bar{U} \neq \emptyset\}, \\ \mathcal{K}_1 &= \{I: I \text{ is a component of } K_1 \text{ and } I \cap \bar{V} \neq \emptyset\}, \\ \mathcal{K}_2 &= \{I: I \text{ is a component of } K_2 \text{ and } I \cap \bar{V} \neq \emptyset\}.\end{aligned}$$

For $i = 1, 2$, let $H'_i = \mathcal{H}_i^*$ and $K'_i = \mathcal{K}_i^*$. These four sets are closed. Clearly $H'_i \subset H_i$ and $K'_i \subset K_i$. Recalling that H_1 and H_2 are mutually separated and that K_1 and K_2 are mutually separated, we see that $H'_1 \cap H'_2 = \emptyset$ and $K'_1 \cap K'_2 = \emptyset$. Since T_1 intersects $\bar{U}$ and $T_1 \subset H_1$, it follows that T_1 is a subcontinuum of a continuum in $\mathcal{H}_1$. Thus $T_1 \subset H'_1$. Consequently, T_1 and H'_2 are mutually exclusive. This, together with a similar argument with respect to K_1 , gives us that T_1 and $H'_2 \cup K'_2$ are mutually exclusive. Similarly, T_2 and $H'_1 \cup K'_1$ are mutually exclusive.

Suppose that p is in $X \setminus (U \cup V)$. Let N denote the component of $X \setminus (U \cup V)$ which contains p . Suppose that p is in $H'_1 \cup H'_2$. Let I be the component of $X \setminus V$ which contains p . The component of $I \setminus U$ which contains p is N . From the definition of H'_1 and H'_2 , we know that I intersects $\bar{U}$, so it follows from [6, Theorem 51, p. 18] that the component of $I \setminus \bar{U}$ which contains p has a limit point in $\bar{U}$. Therefore N intersects $\bar{U}$. Similarly, if p is in $K'_1 \cup K'_2$, then N intersects $\bar{V}$.

Suppose that $H'_1 \cup K'_1$ and $H'_2 \cup K'_2$ are not mutually exclusive. Then either H'_1 intersects K'_2 or H'_2 intersects K'_1 . Supposing the former, let p be a point in $H'_1 \cap K'_2$. Since $H'_1 \cap V = \emptyset$ and $K'_2 \cap U = \emptyset$, we have that p is in $X \setminus (U \cup V)$. Then the component N of $X \setminus (U \cup V)$ which contains p intersects both $\bar{U}$ and $\bar{V}$. Consequently either $N = T_1$ or $N = T_2$. This is contradictory, since $T_1 \cap K'_2 = \emptyset$ and $T_2 \cap H'_1 = \emptyset$. Thus, H'_1 and K'_2 are mutually exclusive. The argument that H'_2 and K'_1 are mutually exclusive is similar.

Therefore $H'_1 \cup K'_1$ and $H'_2 \cup K'_2$ are mutually exclusive closed subsets of X containing T_1 and T_2 respectively. We should note here that X is not necessarily the union of $H'_1 \cup K'_1$ and $H'_2 \cup K'_2$, since $X \setminus (U \cup V)$ may have components which intersect neither $\bar{U}$ nor $\bar{V}$.

Suppose that X is not the union of two mutually exclusive closed sets, one containing $H'_1 \cup K'_1$, the other containing $H'_2 \cup K'_2$. Then there is a continuum $I \subset X$ which is irreducible between $H'_1 \cup K'_1$ and $H'_2 \cup K'_2$. Let p be a point in $I \setminus (H'_1 \cup K'_1 \cup H'_2 \cup K'_2)$. Since $U \subset H'_1 \cup H'_2$ and $V \subset K'_1 \cup K'_2$, p is in $X \setminus (U \cup V)$.

Suppose $I \subset X \setminus (U \cup V)$. Let q be a point in $I \cap (H'_1 \cup K'_1)$. Suppose q is in H'_1 . Let J be the component of H_1 which contains q . Since $J \cup I$ is a connected subset of $X \setminus V$ which contains a point of H_1 , and H_1 and H_2 form a separation of $X \setminus V$, it follows that $J \cup I \subset H_1$. Since q is in H'_1 , J intersects $\bar{U}$, and we have that $J \cup I \subset H'_1$. This puts p in H'_1 , a contradiction. We reach a similar contradiction if we suppose that q is in K'_1 . Therefore $I \subset X \setminus (U \cup V)$.

Suppose that I intersects U . Let J be the component of $I \setminus (U \cup V)$ which contains p . Then J intersects $\bar{U} \cup \bar{V}$. Suppose that J intersects $\bar{U}$. Since $X \setminus V = H_1 \cup H_2$, either $J \subset H_1$ or $J \subset H_2$. If $J \subset H_1$ then $J \subset H'_1$ so p is in H'_1 , a contradiction. If $J \subset H_2$, then $J \subset H'_2$, also a contradiction. If J intersects $\bar{V}$, on the other hand, similar contradictions arise.

Therefore X is the union of two mutually separated sets, one containing $H'_1 \cup K'_1$, the other containing $H'_2 \cup K'_2$. Since $T_1 \subset H'_1 \cup K'_1$ and $T_2 \subset H'_2 \cup K'_2$ we have the conclusion of the theorem.

The three cases. We now return to the proof of Theorem 1. The notation is that established in the first part of the proof.

For N in C_0 let $\lambda(N)$ be the component of $f^{-1}(W_0)$ which contains N . If N is in C_0 , it follows that $\lambda(N)$ is in C and thus $\lambda(N)$ intersects $\bar{V}_b$.

Case I. Suppose that C_0 has only one member, T . Let H be the component of $f^{-1}(A \cup C_0)$ which contains a' and let K be the component of $f^{-1}(A_0 \cup C)$ which contains c' . By Lemma 2, H includes a component of $X_{ac} \setminus (V_a \cup V_c)$ which intersects both $\bar{V}_a$ and $\bar{V}_c$. Since T is the only such component, $T \subset H$. That $T \subset K$ follows by a similar argument.

Thus H , K , and $\lambda(T)$ are three continua with a point in common (any point of T), no one of which is contained in the union of the other two. Thus, by Sorgenfrey's Theorem [7, Theorem 1.8, p. 443], $\lambda(T) \cup H \cup K$ contains a triod. This contradiction completes Case I.

Case II. Suppose that C_0 has just two members, T_1 and T_2 . From Theorem 2 it follows that either both T_1 and T_2 are included in the same component of $X_{ac} \setminus V_a$ or both T_1 and T_2 are included in the same component of $X_{ac} \setminus V_c$. We will suppose the former. A similar argument holds in case the latter holds. Let H be the component of $f^{-1}(A \cup C_0)$ which contains a' and let K be the component of $f^{-1}(A_0 \cup C)$ which contains c' . By Lemma 2, K includes a component of $X_{ac} \setminus (V_a \cup V_c)$ which intersects both $\bar{V}_a$ and $\bar{V}_c$. Thus either $T_1 \subset K$ or $T_2 \subset K$ and hence $T_1 \subset K$ and $T_2 \subset K$. Also by Lemma 2 either $T_1 \subset H$ or $T_2 \subset H$. Let n be the integer in $\{1, 2\}$ such that $T_n \subset H$. Now $\lambda(T_n)$ is a continuum which intersects $\bar{V}_b$. Thus H , K , and $\lambda(T_n)$ are three continua with a point in common, no one of which is contained in the union of the other two. By [7, Theorem 1.8, p. 443] $H \cup K \cup \lambda(T_n)$ contains a triod. This is a contradiction to the atriodicity of X .

Case III. Suppose that C_0 has three members, T_1 , T_2 , and T_3 . By [6, Theorem 52, p. 21], there exist open subsets, O_1 , O_2 , and O_3 , of X , such that $\bar{O}_i$ and $\bar{O}_j$ are mutually exclusive for $i \neq j$, and such that, for $i = 1, 2, 3$, $T_i \subset O_i$, $f^{-1}(A_0 \cup C_0)$ and $\text{bdry}(O_i)$ are mutually exclusive, and O_i and $f^{-1}(U_b)$ are mutually exclusive. For each i , let L_i be the closure of the component of $\lambda(T_i) \cap O_i$ which includes T_i . There is a point p_i of L_i in the boundary of O_i by [6, Theorem 50, p. 18]. By the choice of O_i , p_i does not belong to $f^{-1}(A_0 \cup C_0)$, and thus p_i does not belong to X_{ac} . Since $\bar{O}_i$ and $\bar{O}_j$ are mutually exclusive for $i \neq j$, it follows that p_i is not in L_j .

Let $K_i = L_i \cup X_{ac}$ for $i = 1, 2, 3$. Then K_i is a continuum and, if $i \neq j$, $K_i \cap K_j = X_{ac}$, a proper subcontinuum of both K_i and K_j . Thus $K_1 \cup K_2 \cup K_3$ is a triod.

Applications and questions. Grispolakis and Tymchatyn have proven [2, Corollaries 4.5 and 4.6, pp. 352–353] that semiconfluent mappings preserve one-dimensional acyclic continua and tree-like continua. They have recently proven [3, Corollary 3.4] that acyclic atriodic continua are one dimensional (the author thanks the referee for bringing this paper to his attention). When we combine these results

with Theorem 1 we see that semiconfluent mappings preserve important subclasses of the class of atriodic continua.

COROLLARY 1. *Semiconfluent mappings preserve atriodic acyclic continua.*

COROLLARY 2. *Semiconfluent mappings preserve atriodic tree-like continua.*

We close with some questions which are motivated by Theorem 1 and its proof. Suppose that X and Y are continua and that f is a semiconfluent mapping of X onto Y . Suppose $W \subset Y$ is a triod. In the proof of Theorem 1 we construct a triod $T \subset X$ such that $f(T) \subset W$. Can a triod $T \subset X$ be constructed such that $f(T) = W$? One might also ask whether Theorem 1 can be generalized to n -ods. That is, if L is an n -od in Y , is there an n -od $K \subset X$ such that $f(K) \subset L$ (or $f(K) = L$)?

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF RICHMOND, RICHMOND, VIRGINIA 23173