SIMPLE EXCEPTIONAL 16-DIMENSIONAL JORDAN TRIPLE SYSTEMS

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ABSTRACT. We prove that every simple exceptional 16-dimensional Jordan triple system is a subtriple of an exceptional Jordan algebra.

We prove a reverse triality principle which, in the language of Jordan pairs, states that every (semi)linear automorphism of a 16-dimensional exceptional Jordan pair can be extended to an automorphism of the 27-dimensional Jordan pair of the exceptional reduced Jordan algebra $H_3(\theta)$, θ an octonion algebra. As an immediate application, we obtain that every 16-dimensional exceptional Jordan triple system is a subtriple of a 27-dimensional exceptional Jordan triple system. We will use the notations of [2] for Jordan pairs and those of [1, 2] for the exceptional reduced Jordan algebra. We make no assumption about our base field F . A similar, but different, situation has been examined in [4].

We recall some definitions: Let θ be an octonion (Cayley-Dickson) algebra over F . We have an involution $a \rightarrow \bar{a}$, trace $\text{tr}(a) = a + \bar{a}$, and norm $n(a) = a\bar{a}$, $a \in \theta$. Let $H_3(\theta)$ be the set of 3 by 3 matrices with entries in θ that are symmetric under $X \rightarrow {}^t\bar{X}$ and have main diagonal entries in F . Every $X \in H_3(\theta)$ has the form

$$\sum_{i=1}^3 \alpha_i [ii] + \sum_{(i,j,k) \text{ cyclic}} a_i [jk] \quad \text{with } \alpha_i \in F, a_i \in \theta,$$

where (i, j, k) is a cyclic permutation of $(1, 2, 3)$, $a[ij] = aE_{ij} + \bar{a}E_{ij}$, $\alpha[ii] = \alpha E_{ii}$, in terms of the matrix units E_{ij} . $H_3(\theta)$ is a quadratic Jordan algebra with norm

$$\begin{aligned} N \left(\sum \alpha_i [ii] + \sum a_i [jk] \right) \\ = \alpha_1 \alpha_2 \alpha_3 - \alpha_1 n(a_1) - \alpha_2 n(a_2) - \alpha_3 n(a_3) + \text{tr}(a_1 a_2 a_3). \end{aligned}$$

Two copies of $H_3(\theta)$ define the associated Jordan pair $\underline{V}$. A vector space semi-automorphism $\eta: H_3(\theta) \rightarrow H_3(\theta)$ is the plus part of a Jordan pair automorphism of $\underline{V}$ iff it is a semisimilarity of $H_3(\theta)$, i.e., fulfills

$$N(\eta(X)) = p\tau(N(X))$$

with $p \in F^*$, $\tau \in \text{Gal}(F)$.

An isomorphism $\phi: \underline{V} \rightarrow \underline{V}^{\text{op}}$ with $\phi^2 = \text{id}$ is called involution of $\underline{V}$. ($\underline{V}^{\text{op}}$ is the Jordan pair with plus and minus parts exchanged.) An involution of $\underline{V}$ (up to conjugacy by an automorphism) corresponds to a Jordan triple structure (up to isomorphism) of $\underline{V}$ [3, 1.13; 6].

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Let E_1 denote $(1[11], 1[11]) \in \underline{V} = (H_3(\theta), H_3(\theta))$. The Jordan pair $[M_{1,2}(\theta), M_{2,1}(\theta)]$ is isomorphic to the Peirce 1-space $\underline{V}_1(E_1)$ of $\underline{V}$, where $\underline{V}(E_1)^\sigma = \{a_2[31] + a_3[12] \mid a_2, a_3 \in \theta\}$, $\sigma = \pm 1$.

Our main result is the following theorem.

THEOREM. *Every Jordan pair automorphism $\Phi = (\Phi^+, \Phi^-)$ of $\underline{V}_1(E_1)$ can be extended to a Jordan pair automorphism of $\underline{V}$ fixing E_1 .*

Of course, every automorphism of $\underline{V}$ which leaves $\underline{V}_2(E_1)$ invariant, also leaves $\underline{V}_1(E_1)$ invariant.

PROOF. A quasi-invertible element (x, y) of $\underline{V}_1(E_1)$ remains quasi-invertible in $\underline{V}$, and the corresponding inner automorphism $\beta(x, y)$ of $\underline{V}$ maps E_1 to an idempotent in $V_2(E_1)$ which necessarily is of the form $(\alpha[11], \alpha^{-1}[11])$.

If θ is not split $(1[31], 1[31])$ is a frame of $V_1(E_2)$. Otherwise there exist two orthogonal idempotents e_1, e_2 of θ such that $e_1 + e_2 = 1$, in which case $((e_1[31], e_1[31]), (e_2[31], e_2[31]))$ is a frame of $V_1(E_1)$.

Petersson's Conjugacy Theorem [5] asserts that there exist an inner automorphism of $V_1(E_1)$ which maps the image under Φ of this frame to the frame itself. As inner automorphisms can be lifted we only have to follow Φ by this inner automorphism to be able to assume $\Phi^\sigma(1[31]) = 1[31]$. Then Φ leaves invariant the Peirce decomposition of $\underline{V}_1(E_1)$ with respect to $(1[31], 1[31])$. We define coordinate maps by

$$\phi_{12}^\sigma(x)[12] = \Phi^\sigma(x[12]), \quad \phi_{31}^\sigma(y)[31] = \Phi^\sigma(y[31]).$$

We define

$$p^\sigma = \phi_{12}^\sigma(1) \in \theta^*.$$

We use the Jordan matrix identities

$$(JM1) \quad Q_{x[ij]}^\sigma y[ij] = x\bar{y}x[ij],$$

$$(JM2) \quad \{x[ij], y[ji], z[ik]\} = x(yx)[ik]$$

for $(i, j), (i, k) = (1, 2), (3, 1)$ to obtain

$$(1) \quad \phi_{ij}^\sigma(x\bar{y}x) = \phi_{ij}^\sigma(x)\overline{\phi_{ij}^{-\sigma}(y)}\phi_{ij}^\sigma(x),$$

where (i, j) is either $(1, 2)$ or $(3, 1)$,

$$(2) \quad \phi_{12}^\sigma(x)(\overline{\phi_{12}^{-\sigma}(y)}\phi_{31}^\sigma(z)) = \overline{\phi_{31}^\sigma((zy)\bar{x})},$$

$$(3) \quad \overline{\phi_{31}^\sigma(x)}(\phi_{31}^{-\sigma}(y)\phi_{12}^\sigma(z)) = \phi_{12}^\sigma(\bar{x}(yz)).$$

Let us abbreviate ϕ_{31}^σ by ϕ^σ . Recall $\phi^\sigma(1) = 1$. A consequence of (1) is

$$(4) \quad \phi^{-\sigma}(x) = \overline{\phi^\sigma(\bar{x})} \quad \text{and} \quad p^\sigma \overline{p^{-\sigma}} = 1.$$

Setting $x = z = 1$ in (3) yields

$$(5) \quad \phi^{-\sigma}(x)p^\sigma = \phi_{12}^\sigma(x).$$

If we let $z = 1$ in (3), using (4) and (5), we obtain the crucial identity

$$(6) \quad \phi^{-\sigma}(x)(\phi^{-\sigma}(y)p^\sigma) = \phi^{-\sigma}(xy)p^\sigma.$$

We let $z = 1$ in (2) to get

$$(7) \quad \phi^\sigma(xy) = (\phi^\sigma(x)p^{-\sigma})(\overline{p^\sigma}\phi^\sigma(y)).$$

If (6) holds for one σ , we can use (4) and (5) to define an automorphism of $\underline{V}_1(E_1)$. Indeed, the identity $(ac)(\bar{c}b) = (a(bc))\bar{c}$ for octonion algebras implies that (6) and (7) are equivalent. This enables us to show by a straightforward calculation that (6) for one σ implies (6) for $-\sigma$. By using the octonion identity $(a(bc))(c^{-1}a) = aba$ we may prove that (6) implies (1), (2), and (3). (Both octonion identities may be proved by explicit calculation in the Cayley-Dickson algebra.)

Not let us assume that Φ is τ -linear, $\tau \in \text{Aut}(F)$. We extend Φ by defining

$$\begin{aligned}\Phi^\sigma(x[11]) &= \tau(x)[11], & \Phi^\sigma(x[22]) &= \tau(x)n(p^\sigma)[22], \\ \Phi^\sigma(x[33]) &= \tau(x)[33], & \Phi^\sigma(x[23]) &= \overline{p^\sigma}\phi^{-\sigma}(x)[23].\end{aligned}$$

In order to show that Φ is an automorphism of $\underline{V}$, it suffices to show that Φ^σ is an element of the structure group of $H_3(\theta)$ [3, 1.8] or equivalently a semisimilarity of $H_3(\theta)$ [1, p. 10], i.e.

$$N(\Phi^\sigma(X)) = n(p^\sigma)\tau(N(X)) \quad \text{for } X \in H_3(\theta).$$

In order to prove this we first note that (1) implies $\phi^\sigma(x^2) = (\phi^\sigma(x))^2$; hence,

$$\text{tr}(\phi^\sigma(x))\phi^\sigma(x) - n(\phi^\sigma(x)) = \phi^\sigma(\text{tr}(x)x - n(x)) = \tau(\text{tr}(x))\phi^\sigma(x) - \tau(n(x)).$$

This implies $\text{tr}(\phi^\sigma(x)) = \tau(\text{tr}(x))$ and $\tau(n(x)) = n(\phi^\sigma(x))$. Therefore we have only to show

$$\text{tr}[(\overline{p^\sigma}\phi^{-\sigma}(a_1))(\phi^\sigma(a_2))(\phi^{-\sigma}(a_3)p^\sigma)] = n(p^\sigma)\tau(\text{tr}(a_1a_2a_3)).$$

This is proven by the following calculation, in which we use the linearity as well as the associativity of the trace, which states that $\text{tr}(a(bc)) = \text{tr}((ab)c)$ for $a, b, c \in \theta$:

$$\begin{aligned}& \text{tr}[(\overline{p^\sigma}\phi^{-\sigma}(a_1))(\phi^\sigma(a_2))(\phi^{-\sigma}(a_3)p^\sigma)] \\&= \text{tr}[(\overline{p^\sigma}\phi^{-\sigma}(a_1))(\text{tr}(\phi^\sigma(a_2)) - \overline{\phi^\sigma(a_2)})(\phi^{-\sigma}(a_3)p^\sigma)] \\&= \text{tr}[(\overline{p^\sigma}\phi^{-\sigma}(a_1))(\text{tr}(\phi^\sigma(a_2)))(\phi^{-\sigma}(a_3)p^\sigma)] \\&\quad - \text{tr}[(\overline{p^\sigma}\phi^{-\sigma}(a_1))(\overline{\phi^\sigma(a_2)})(\phi^{-\sigma}(a_3)p^\sigma)] \\&= \text{tr}[\phi^\sigma(a_2)] \text{tr}[(\overline{p^\sigma}\phi^{-\sigma}(a_1))(\phi^{-\sigma}(a_3)p^\sigma)] \\&\quad - \text{tr}[(\overline{p^\sigma}\phi^{-\sigma}(a_1))(\phi^{-\sigma}(\overline{a_2}))(\phi^{-\sigma}(a_3)p^\sigma)] \quad \text{by (4)} \\&= \text{tr}[\phi^\sigma(a_2)] \text{tr}[(\overline{p^\sigma}\phi^{-\sigma}(a_1))(\phi^{-\sigma}(a_3)p^\sigma)] \\&\quad - \text{tr}[(\overline{p^\sigma}\phi^{-\sigma}(a_1))(\phi^{-\sigma}(\overline{a_2}a_3)p^\sigma)] \quad \text{by (6)} \\&= \text{tr}[\phi^\sigma(a_2)]n(p^\sigma) \text{tr}[(\phi^{-\sigma}(a_3)p^\sigma)(\overline{p^\sigma}\phi^{-\sigma}(a_1))] \\&\quad - n(p^\sigma) \text{tr}[(\overline{p^\sigma}\phi^{-\sigma}(a_1))(\phi^{-\sigma}(\overline{a_2}a_3)p^\sigma)] \quad \text{by (4)} \\&= n(p^\sigma) \text{tr}[\phi^\sigma(a_2)] \text{tr}[\phi^{-\sigma}(a_3a_1)] - n(p^\sigma) \text{tr}[\phi^{-\sigma}(a_1(\overline{a_2}a_3))] \\&= n(p^\sigma)\tau(\text{tr}(a_2) \text{tr}(a_3a_1)) - n(p^\sigma)(\tau(\text{tr}(a_1a_2a_3))) \\&= n(p^\sigma)\tau(\text{tr}(a_3 \text{tr}(a_2)a_1) - \text{tr}(a_1\overline{a_2}a_3)) \\&= n(p^\sigma)\tau(\text{tr}(a_1a_2a_3)).\end{aligned}$$

It is easy to show that a Jordan pair automorphism which is the identity on $\underline{V}_2(E_1)$ and $\underline{V}_1(E_1)$ is the identity on $\underline{V}$ and that the identity on $\underline{V}_1(E_1)$ can be lifted only

to an automorphism given by

$$\begin{aligned} \Phi^\sigma & \left(\sum \alpha_i[ii] + \sum_{(i,j,k) \text{ cyclic}} a_i[jk] \right) \\ & = \alpha_\sigma \alpha_1[11] + \alpha_{-\sigma} \alpha_2[22] + \alpha_{-\sigma} \alpha_3[33] + \alpha_{-\sigma} a_1[23] + a_2[31] + a_3[12], \end{aligned}$$

where $\alpha_\sigma \in F$, $\alpha_{-\sigma} = \alpha_\sigma^{-1}$. This remark concludes the proof of the theorem.

As a consequence of the proof, we note that for linear Φ , (7) implies that ϕ^σ is an element of the structure group of the alternative algebra θ . At the same time, it is an element of the automorphism group of the Jordan algebra $\theta^{(+)}$, and hence an element of $O^1(n)$, the rotation group of n . Every element of $O^1(n)$ can be written as

$$U_{a_1} \circ J \circ \cdots \circ U_{a_{2r}} \circ J,$$

where $J(x) = \bar{x}$, $n(a_1) \cdots n(a_{2r}) = 1$, $U_a(x) = axa$, and can be extended by

$$Q_{[22]+a_1[31]}^{+\sigma} \circ \cdots \circ Q_{[22]+a_{2r}[31]}^{-\sigma}$$

to a Jordan pair automorphism (see [1, Chapter 2]). This remark leads to a shorter proof of our result for linear automorphisms.

We apply our result to Jordan triple systems.

COROLLARY 1. *Every involution of $\underline{V}_1(E_1)$ can be extended to an involution of $\underline{V}$.*

PROOF. Let ε denote the switching involution of $\underline{V}$. If η is an involution of $\underline{V}_1(E_1)$, then $\varepsilon \circ \eta$ is an automorphism of $\underline{V}_1(E_1)$ which can be extended to an automorphism $\varepsilon \circ \hat{\eta}$ which is the identity on $\underline{V}_2(E_1)$. As $\eta^2 = \text{id}$, $(\eta \circ \varepsilon) \circ (\varepsilon \circ \eta) = \text{id}$, hence $(\hat{\eta} \circ \varepsilon)$ is the identity on $\underline{V}_2(E_1)$ and $\underline{V}_1(E_1)$. Therefore $\hat{\eta}^2 = \text{id}$.

COROLLARY 2. *Every simple exceptional 16-dimensional Jordan triple system is a Jordan subtriple system of an exceptional simple reduced Jordan algebra.*

PROOF. A simple exceptional 16-dimensional Jordan triple system is given by an involution of a simple exceptional 16-dimensional Jordan pair.

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