ON THE BAIRE CLASS OF $\mathcal{J}$ -APPROXIMATE DERIVATIVES

EWA ŁAZAROW

ABSTRACT. Basic properties of an $\mathcal{J}$ -approximate derivative which is a category analogue of the approximate derivative were considered in [3]. Using the notion of balanced selection and the result of O'Malley, we shall prove that the $\mathcal{J}$ -approximate derivative is of Baire class one.

Throughout the paper, $\mathcal{B}$ will denote the family of all subsets of R having the Baire property and $\mathcal{J}$ the σ -ideal of sets of the first category. For $a \in R$ and $A \subset R$ we denote $a \cdot A = \{ax : x \in A\}$ and $A - a = \{x - a : x \in A\}$. Recall [4] that 0 is an $\mathcal{J}$ -density point of a set $A \in \mathcal{B}$ if and only if

$$\chi_{n \cdot A \cap [-1, 1]} \xrightarrow{\mathcal{J}} 1 \quad \text{as } n \rightarrow \infty,$$

i.e. if and only if for every increasing sequence $\{n_m\}_{m \in N}$ of natural numbers there exists a subsequence $\{n_{m_p}\}_{p \in N}$ such that

$$\chi_{n_{m_p} \cdot A \cap [-1, 1]} \rightarrow 1 \quad \text{as } p \rightarrow \infty$$

except for a set belonging to $\mathcal{J}$. A point x_0 is an $\mathcal{J}$ -density point of $A \in \mathcal{B}$ if and only if 0 is an $\mathcal{J}$ -density point of $A - x_0$. A point x_0 is an $\mathcal{J}$ -dispersion point of $A \in \mathcal{B}$ if and only if x_0 is an $\mathcal{J}$ -density point of $R \setminus A$. The notions of right-hand and left-hand $\mathcal{J}$ -density points are defined in an obvious manner.

DEFINITION. Let F be any finite function defined in some neighborhood of x_0 , and having there the Baire property, and let

$$C(x, x_0) = \frac{F(x) - F(x_0)}{x - x_0} \quad \text{for } x \neq x_0.$$

We define the $\mathcal{J}$ -approximate upper derivative ($\bar{F}'_{\mathcal{J}\text{-ap}}(x_0)$) as the greatest lower bound of the set

$$\{\alpha : \{x : C(x, x_0) > \alpha\} \text{ has } x_0 \text{ as an } \mathcal{J}\text{-dispersion point}\}.$$

The $\mathcal{J}$ -approximate lower derivatives ($\underline{F}'_{\mathcal{J}\text{-ap}}(x_0)$) are defined similarly. If $\bar{F}'_{\mathcal{J}\text{-ap}}(x_0) = \underline{F}'_{\mathcal{J}\text{-ap}}(x_0)$, their common value is called the $\mathcal{J}$ -approximate derivative of F at x_0 ($F'_{\mathcal{J}\text{-ap}}(x_0)$).

To prove the above-mentioned result, we need a preliminary lemma and some theorems.

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THEOREM PWW [4]. A point x_0 is an $\mathcal{J}$ -density point of the set $A \in \mathcal{B}$ if and only if, for every increasing sequence $\{t_n\}_{n \in \mathbb{N}}$ of positive real numbers tending to infinity, there exists a subsequence $\{t_{n_m}\}_{m \in \mathbb{N}}$ such that

$$\chi_{(t_{n_m} \cdot (A - x_0)) \cap [-1, 1]} \rightarrow 1 \quad \text{as } m \rightarrow \infty$$

except for a set belonging to $\mathcal{J}$.

LEMMA [4]. If 0 is an $\mathcal{J}$ -density point of the set A , then, for every natural number n , there exists a real number $\delta_n > 0$ such that, for each $h \in (0, \delta_n)$ and for each natural number k fulfilling the inequality $-n \leq k \leq n - 1$, we have

$$A \in \left[\frac{k}{n}h, \frac{k+1}{n}h \right] \neq \emptyset.$$

THEOREM 1. A point 0 is an $\mathcal{J}$ -dispersion point of the open set G if and only if, for every natural number n , there exist a natural number k and a real number $\delta > 0$ such that, for each $h \in (0, \delta)$ and for each $i \in \{1, \dots, n\}$, there exist two natural numbers $j_r, j_l \in \{1, \dots, k\}$ such that

$$G \cap \left(\left(\frac{i-1}{n} + \frac{j_r-1}{n \cdot k} \right)h, \left(\frac{i-1}{n} + \frac{j_r}{n \cdot k} \right)h \right) = \emptyset$$

and

$$G \cap \left(-\left(\frac{i-1}{n} + \frac{j_l}{n \cdot k} \right)h, -\left(\frac{i-1}{n} + \frac{j_l-1}{n \cdot k} \right)h \right) = \emptyset.$$

PROOF. *Necessity.* We shall first prove necessity for a right-hand $\mathcal{J}$ -dispersion. Suppose that it is not the case. Then there exists a natural number n_0 such that, for each $k \in \mathbb{N}$ and $\delta_k = 1/k$, there exist $h_k \in (0, 1/k)$ and $i_k \in \{1, \dots, n_0\}$ such that, for each $j \in \{1, \dots, k\}$,

$$G \in \left(\left(\frac{i_k-1}{n_0} + \frac{j-1}{n_0 k} \right)h_k, \left(\frac{i_k-1}{n_0} + \frac{j}{n_0 k} \right)h_k \right) \neq \emptyset.$$

Let $\{h_{k_m}\}_{m \in \mathbb{N}}$ be a subsequence of $\{h_k\}$ such that the number i_{k_m} is common for all m (for example, for every m , $i_{k_m} = i_0$) and let $t_{k_m} = 1/h_{k_m}$ for all $m \in \mathbb{N}$. Then $\{t_{k_m}\}_{m \in \mathbb{N}}$ tends to infinity. Let $\{t_{k_{m_s}}\}$ be any subsequence of $\{t_{k_m}\}$. For every natural number p , the set

$$\bigcup_{s=p}^{\infty} \left(t_{k_{m_s}} \cdot G \cap \left(\frac{i_0-1}{n_0}, \frac{i_0}{n_0} \right) \right)$$

is open and dense on $[(i_0 - 1)/n_0, i_0/n_0]$; so

$$\bigcap_{p=1}^{\infty} \bigcup_{s=p}^{\infty} \left(t_{k_{m_s}} \cdot G \cap \left[\frac{i_0-1}{n_0}, \frac{i_0}{n_0} \right] \right)$$

is residual on $[(i_0 - 1)/n_0, i_0/n_0]$. Consequently,

$$\limsup_s (t_{k_{m_s}} \cdot G \cap [-1, 1]) \supset \bigcap_{p=1}^{\infty} \bigcup_{s=p}^{\infty} \left(t_{k_{m_s}} \cdot G \cap \left[\frac{i_0-1}{n_0}, \frac{i_0}{n_0} \right] \right) \notin \mathcal{J},$$

implying that $\{\chi_{(t_{k_m} \cdot G \cap [0,1])}\}$ does not converge to 0 except for a set belonging to $\mathcal{J}$. Since this is true for each subsequence of $\{t_{k_m}\}$, Theorem PWV implies that 0 is not a right-hand $\mathcal{A}$ -dispersion point of G , which is a contradiction.

So, we have proved that, for every natural n there exist a natural number k_r and a positive real number δ_r such that, for each $h \in (0, \delta_r)$ and for each $i \in \{1, \dots, n\}$, there exists $j_r \in \{1, \dots, k_r\}$ such that

$$G \cap \left(\left(\frac{i-1}{n} + \frac{j_r-1}{n \cdot k_r} \right) h, \left(\frac{i-1}{n} + \frac{j_r}{n \cdot k_r} \right) h \right) = \emptyset.$$

It is obvious that if we take $k > 2k_r$ and $0 < \delta < \delta_r$, then, for each $h \in (0, \delta)$ and for each $i \in \{1, \dots, n\}$, we can also find $j \in \{1, \dots, k\}$ for which

$$G \cap \left(\left(\frac{i-1}{n} + \frac{j-1}{n \cdot k} \right) h, \left(\frac{i-1}{n} + \frac{j}{n \cdot k} \right) h \right) = \emptyset.$$

From the fact that 0 is a left-hand $\mathcal{A}$ -dispersion point of G we conclude in a similar way that, for every natural n , there exist a natural number k_l and a positive real number δ_l such that, for each $h \in (0, \delta_l)$ and $i \in \{1, \dots, n\}$, there exists $j_l \in \{1, \dots, k_l\}$ such that

$$G \cap \left(-\left(\frac{i}{n} + \frac{j_l}{n \cdot k_l} \right) h, -\left(\frac{i}{n} + \frac{j_l-1}{n \cdot k_l} \right) h \right) = \emptyset.$$

Let n be fixed. Take $k = 2 \max(k_r, k_l)$, $\delta = \min(\delta_r, \delta_l)$. From the above remark it follows that k and δ fulfill all our requirements.

Sufficiency. Let $\{t_p\}_{p \in N}$ be an arbitrary increasing sequence of positive real numbers tending to infinity.

The subsequence $\{t_{p_s}\}_{s \in N}$ will be defined by induction. For $n = 1$, there exist k_1 and $\delta_1 > 0$ such that, for each $h \in (0, \delta_1)$ and $i = 1$, there exists $j = j(h, 1) \in \{1, \dots, k_1\}$ such that

$$G \cap \left(\frac{j-1}{k_1} h, \frac{j}{k_1} h \right) = \emptyset.$$

Let $\{t_{\alpha_1(s)}\}_{s \in N}$ be a subsequence of $\{t_p\}_{p \in N}$ such that, for each $s \in N$, $t_{\alpha_1(s)} > 1/\delta_1$ and the number $j(1/t_{\alpha_1(s)}, 1) = j_{11}$ is common for all $s \in N$. We define $t_{p_1} = t_{\alpha_1(1)}$.

Assume the sequence $\{t_{\alpha_{n-1}(s)}\}_{s \in N}$ and $t_{p_{n-1}} = t_{\alpha_{n-1}(1)}$ to be defined. For a natural number n , there exist k_n and $\delta_n > 0$ such that, for each $h \in (0, \delta_n)$ and $i = \{1, \dots, n\}$, there exists $j = j(h, i) \in \{1, \dots, k_n\}$ such that

$$G \cap \left(\left(\frac{i-1}{n} + \frac{j-1}{n \cdot k_n} \right) h, \left(\frac{i-1}{n} + \frac{j}{n \cdot k_n} \right) h \right) = \emptyset.$$

Let $\{t_{\alpha_n(s)}\}_{s \in N}$ be a subsequence of $\{t_{\alpha_{n-1}(s)}\}_{s \in N}$ such that, for each $s \in N$, $t_{\alpha_n(s)} > 1/\delta_n$ and $j(1/t_{\alpha_n(s)}, 1) = j_{n1}, \dots, j(1/t_{\alpha_n(s)}, n) = j_{nn}$ are common for all $s \in N$. Put $t_{p_n} = t_{\alpha_n(1)}$.

We shall show that $\{x: \chi_{(t_{p_n} \cdot G \cap [0,1])} \rightarrow 0\} \in \mathcal{J}$. Let (a, b) be an arbitrary subset of $[0, 1]$. Then there exist a natural number n_0 and $i_0 \in \{1, \dots, n_0\}$ such that $[(i_0 - 1)/n_0, i_0/n_0] \subset (a, b)$. We shall consider a sequence $\{t_{\alpha_{n_0}(s)}\}_{s \in N}$ and a

natural number k_{n_0} corresponding to n_0 . Then, for each $n \geq n_0$, $t_{p_n} \in \{t_{\alpha_{n_0}(s)}\}_{s \in N}$. Therefore, for each $n > n_0$, there exists $j = j_{n_0, i_0}$ such that

$$G \cap \left(\left(\frac{i_0 - 1}{n_0} + \frac{j - 1}{n_0 \cdot k_{n_0}} \right) \frac{1}{t_{p_n}}, \left(\frac{i_0 - 1}{n_0} + \frac{j}{n_0 \cdot k_{n_0}} \right) \frac{1}{t_{p_n}} \right) = \emptyset.$$

Let

$$(a_1, b_1) = \left(\frac{i_0 - 1}{n_0} + \frac{j}{n_0 \cdot k_{n_0}}, \frac{i_0 - 1}{n_0} + \frac{j}{n_0 \cdot k_{n_0}} \right) \subset (a, b).$$

Then, for each $n \geq n_0$, we have that

$$\emptyset = G \cap \left(a_1 \cdot \frac{1}{t_{p_n}}, b_1 \cdot \frac{1}{t_{p_n}} \right) = \frac{1}{t_{p_n}} (t_{p_n} \cdot G \cap (a_1, b_1)),$$

so

$$(a_1, b_1) \subset [0, 1] \setminus (t_{p_n} \cdot G \cap [0, 1]).$$

Therefore

$$(a_1, b_1) \subset \bigcup_{r=1}^{\infty} \bigcap_{n=r}^{\infty} ([0, 1] \setminus (t_{p_n} \cdot G \cap [0, 1]))$$

and $\limsup_r (t_{p_r} \cdot G \cap [0, 1])$ is a nowhere dense set. Thus

$$\{x: \chi_{(t_{p_n} \cdot G \cap [0, 1])} \not\rightarrow 0\} \in \mathcal{J}$$

and, by Theorem PWW, we have that 0 is the right-hand $\mathcal{J}$ -dispersion point of G .

In a similar way we can show that 0 is the left-hand $\mathcal{J}$ -dispersion point of G . The theorem is proved.

We shall use the above theorem and lemma for each $x \in R$ by translating the set, if necessary.

In [3] we proved that if f has an $\mathcal{J}$ -approximate derivative at all $x \in [0, 1]$, then there exists a selection S such that f has a selective derivative at all $x \in [0, 1]$. The concept of a function f having a selective derivative was introduced and examined in [1] by R. J. O'Malley. One of the contrasts between a selective derivative and an ordinary derivative is the fact that certain selective derivatives are strictly of Baire class two. So, in [3] we proved that some $\mathcal{J}$ -approximate derivatives are of Baire class two.

R. J. O'Malley proved the following theorem [2].

THEOREM. *If $f: R \rightarrow R$ has a selective derivative $g(x)$ with respect to a selection S which is balanced relative to functions $\alpha(x), \delta(x)$, then g is of Baire class one.*

A selection S can be thought of as an interval function defined on the class of all nondegenerate closed subintervals $[a, b]$ of R , with the property that, for each $[a, b]$, $a < s(a, b) < b$. If we have the convention: for a fixed interval $[a, b]$ and for a fixed number $\alpha \in (0, 1)$, by the α -interval of $[a, b]$ we mean the interval centered in $[a, b]$ of length $\alpha(b - a)$, then a selection S is said to be balanced if there are two functions $\alpha(x)$ and $\delta(x)$ such that $\alpha(x) > 0$, $\delta(x) > 0$, and if $I = [a, b]$ is any interval having x as one endpoint and $b - a < \delta(x)$, then $s[a, b]$ is in the $\alpha(x)$ -interval of $[a, b]$.

By the above, we shall show that the $\mathcal{A}$ -approximate derivatives are of Baire class one. The proof of that fact is similar to the proof of [3, Theorem 2] but, for the reader's convenience, we provide it.

THEOREM 2. *If $f: [0, 1] \rightarrow \mathbb{R}$ has a finite $\mathcal{A}$ -approximate derivative $f'_{\mathcal{A}\text{-ap}}(x)$ at all $x \in [0, 1]$, then $f'_{\mathcal{A}\text{-ap}}(x)$ is of Baire class one.*

PROOF. Let $\varepsilon > 0$ and $c \in [0, 1]$. We denote

$$C(\varepsilon) = \left\{ t: f'_{\mathcal{A}\text{-ap}}(c) - \varepsilon \leq \frac{f(t) - f(c)}{t - c} \leq f'_{\mathcal{A}\text{-ap}}(c) + \varepsilon \right\}.$$

By assumption, c is an $\mathcal{A}$ -density point of $C(\varepsilon)$ and $C(\varepsilon) \in \mathcal{B}$, $[0, 1] \setminus C(\varepsilon) \in \mathcal{B}$. These last facts follow from the simple observation that $\mathcal{A}$ -approximate differentiability at x_0 implies $\mathcal{A}$ -approximate continuity at x_0 , and an $\mathcal{A}$ -approximately continuous function has the Baire property (because it belongs to the first class of Baire [4]). Then $C(\varepsilon) = G_c \triangle P_1$ and $[0, 1] \setminus C(\varepsilon) = H_c \triangle P_2$, where G_c , H_c are open and P_1 , P_2 are of the first category.

Since c is an $\mathcal{A}$ -density point of G_c and an $\mathcal{A}$ -dispersion point of H_c , we have

(1) by Theorem 1, for $n = 3$, there exist a natural number $k_c(\varepsilon)$ and a real number $\delta_c(\varepsilon) > 0$ such that, for each $h \in (0, \delta_c(\varepsilon))$, there exist two natural numbers $i_r, i_p \in \{1, \dots, k_c(\varepsilon)\}$ such that

$$H_c \cap \left(c + \left(\frac{1}{3} + \frac{i_r - 1}{3k_c(\varepsilon)} \right) h, c + \left(\frac{1}{3} + \frac{i_r}{3k_c(\varepsilon)} \right) h \right) = \emptyset$$

and

$$H_c \cap \left(c - \left(\frac{2}{3} + \frac{-i_r}{3k_c(\varepsilon)} \right) h, c - \left(\frac{2}{3} + \frac{-i_p - 1}{3k_c(\varepsilon)} \right) h \right) = \emptyset;$$

(2) by the Lemma, for $n = 3k_c(\varepsilon)$, there exists a real number $\delta_{k_c(\varepsilon)} > 0$ such that, for each $h \in (0, \delta_{k_c(\varepsilon)})$ and for each $i \in \{-3k_c(\varepsilon) + 1, \dots, 3k_c(\varepsilon)\}$,

$$G_c \cap \left(c + \frac{i - 1}{3k_c(\varepsilon)} \cdot h, c + \frac{i}{3k_c(\varepsilon)} \cdot h \right) \neq \emptyset.$$

Now, we shall define the family of sets $\{E_{kj}(\varepsilon)\}$ for $k, j \in \mathbb{N}$ and $\varepsilon > 0$. We shall say that $c \in E_{kj}(\varepsilon)$ if and only if $k = k_c(\varepsilon)$ where $k_c(\varepsilon)$ is the number described in (1) and j is a natural number such that $1/j < \min\{\delta_c(\varepsilon), \delta_{k_c(\varepsilon)}\}$ where $\delta_c(\varepsilon)$, $\delta_{k_c(\varepsilon)}$ are the real numbers described in (1) and (2), respectively. Then $[0, 1] = \bigcap_\varepsilon \bigcup_k \bigcup_j E_{kj}(\varepsilon)$ and the sets $E_{kj}(\varepsilon)$ also have the following properties: if $k' \leq k''$ and $j' \leq j''$, then $E_{k',j'}(\varepsilon) \subset E_{k'',j''}(\varepsilon)$, and, for $\varepsilon_1, \varepsilon_2$, if $c \in E_{kj}(\varepsilon_1)$, $d \in E_{kj}(\varepsilon_2)$, and $0 < d - c < 1/j$, then $C(\varepsilon_1) \cap D(\varepsilon_2) \cap (c + \frac{1}{3}(d - c), c + \frac{2}{3}(d - c)) \neq \emptyset$ where

$$D(\varepsilon_2) = \left\{ t: f'_{\mathcal{A}\text{-ap}}(d) - \varepsilon_2 \leq \frac{f(t) - f(d)}{t - d} \leq f'_{\mathcal{A}\text{-ap}}(d) + \varepsilon_2 \right\}.$$

Let $[c, d] \subset [0, 1]$ and $\varepsilon_c = \inf\{\varepsilon: \text{there exist natural numbers } k \text{ and } j \text{ such that } c \in E_{kj}(\varepsilon) \text{ and } |d - c| < 1/j\}$, $\varepsilon_d = \inf\{\varepsilon: \text{there exist natural numbers } k \text{ and } j \text{ such that } d \in E_{kj}(\varepsilon) \text{ and } |d - c| < 1/j\}$.

We need to consider four cases:

(i) $\varepsilon_c < \infty$ and $\varepsilon_d < \infty$. Then there exist natural numbers k_1, k_2, j_1, j_2 such that $c \in E_{k_1, j_1}(\varepsilon_c + |d - c|)$, $d \in E_{k_2, j_2}(\varepsilon_d + |d - c|)$, and $|d - c| < \min(1/j_1, 1/j_2)$. Let $k = \max(k_1, k_2)$ and $j = \max(j_1, j_2)$. Then c and d belong to $E_{kj}(\varepsilon_c + |d - c|)$, and $|d - c| < 1/j$. Thus, for $s([a, b])$ we shall take one point of the set

$$C(\varepsilon_c + |d - c|) \cap D(\varepsilon_d + |d - c|) \cap (c + \tfrac{1}{3}|d - c|, c + \tfrac{2}{3}|d - c|) \neq \emptyset.$$

(ii) $\varepsilon_c < \infty$ and $\varepsilon_d = \infty$. Then there exist natural numbers k and j such that $c \in E_{kj}(\varepsilon_c + |d - c|)$ and $|d - c| < 1/j$. Thus, for $s([c, d])$ we take an arbitrary point of the set $C(\varepsilon_c + |d - c|) \cap (c + \tfrac{1}{3}|d - c|, c + \tfrac{2}{3}|d - c|)$.

(iii) $\varepsilon_c = \infty$ and $\varepsilon_d < \infty$. In a similar way we prove that for $s([c, d])$ we can choose an arbitrary point of the set

$$D(\varepsilon_d + |d - c|) \cap (c + \tfrac{1}{3}|d - c|, c + \tfrac{2}{3}|d - c|) \neq \emptyset.$$

(iv) $\varepsilon_c = \infty$ and $\varepsilon_d = \infty$. Then $s([c, d]) = (c + d)/2$.

We show that this selection gives a selective derivative $sf'(x)$ which equals $f'_{\mathcal{S}\text{-ap}}(x)$ at all x . Let $c \in [0, 1]$ and $\varepsilon > 0$. Then there exist natural numbers k_0 and j_0 such that $c \in E_{k_0 j_0}(\varepsilon/2)$. Let $0 < \delta < \min(1/j_0, \varepsilon/2)$. If $|h| < \delta$, then $\varepsilon/2 \in \{\eta: \text{there exist natural numbers } k \text{ and } j \text{ such that } c \in E_{kj}(\eta) \text{ and } |h| < 1/j\} \neq \emptyset$. Therefore $\varepsilon/2 > \varepsilon_c$ where ε_c stands for the same as before. Let $0 < |h| < \delta$ and $c + h = d$. If there exist $\varepsilon_0 > 0$, k and j such that $|d - c| < 1/j$, and $d \in E_{kj}(\varepsilon_0)$, then $\varepsilon_d < \infty$ and

$$s([c, d]) \in C(\varepsilon_c + |d - c|) \cap D(\varepsilon_d + |d - c|) \cap (c + \tfrac{1}{3}|d - c|, c + \tfrac{2}{3}|d - c|).$$

In the remaining cases, $s([c, d]) \in C(\varepsilon_c + |d - c|) \cap (c + \tfrac{1}{3}|d - c|, c + \tfrac{2}{3}|d - c|)$. Therefore, for all $0 < |h| < \delta$,

$$\left| \frac{f(s([c, c + h])) - f(c)}{s([c, c + h]) - c} - f'_{\mathcal{S}\text{-ap}}(c) \right| \leq \varepsilon_c + |h| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

So $sf'(c) = f'_{\mathcal{S}\text{-ap}}(c)$.

By the above, we have that f has a selective derivative with respect to a selection S which is balanced relative to functions $\alpha(x) = 1/3$ and $\delta(x) = 1$. Therefore $f'_{\mathcal{S}\text{-ap}} \equiv sf'$ is of Baire class one.

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INSTITUTE OF MATHEMATICS, UNIVERSITY OF ŁÓDŹ, UL. BANACHA 22, 90 - 238 ŁÓDŹ, POLAND