ON THE REALIZATION AND CLASSIFICATION OF CYCLIC EXTENSIONS OF POLYNOMIAL ALGEBRAS OVER THE STEENROD ALGEBRA

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ABSTRACT. Suppose $\mathbf{R}^*$ is an unstable algebra over the Steenrod algebra of the form $\mathbf{P}^*(\sqrt[k]{d})$, where $\mathbf{P}^*$ is a polynomial algebra over the Steenrod algebra. If $\mathbf{R}^*$ is integrally closed then $\mathbf{R}^* = P(V)^G_\chi$, where $C \leq GL(V)$ is generated by pseudoreflections and $G_\chi = \ker\{\chi: G \rightarrow \mathbf{F}_p^*\}$ is a character of degree k .

Let $\mathbf{P}^*$ be an unstable polynomial algebra over the Steenrod algebra. By a *cyclic*, or *radical*, *extension of $\mathbf{P}^*$ of degree k* we mean an unstable algebra $\mathbf{R}^*$ over the Steenrod algebra containing $\mathbf{P}^*$ as a subalgebra and closed under the action of the Steenrod algebra such that: $\mathbf{R}^*$ is generated as an algebra by $\mathbf{P}^*$ and a class $f \in \mathbf{R}^*$ such that the minimal polynomial of f over $\mathbf{P}^*$ is of the form $Y^k - d$ for some $d \in \mathbf{P}^*$. In other words $\mathbf{R}^* \cong \mathbf{P}^*(\sqrt[k]{d})$ as algebras over the Steenrod algebra.

Let $V = \bigoplus_n \mathbf{F}_p$, $G \leq GL(V)$ and $\chi: G \rightarrow \mathbf{F}_p^*$ a linear character of G of order k . Following Stanley [6] define

$$P(V)_\chi^G = \{h \in P(V) \mid gh = \chi(g)h \ \forall g \in G\}.$$

This is the $P(V)^G$ -module of χ -relative invariants. Set

$$G_\chi := \ker\{\chi: G \rightarrow \mathbf{F}_p^*\}$$

and note $G/G_\chi \cong \mathbf{Z}/k$. If G is generated by pseudoreflections and of order relatively prime to p then the nonmodular characteristic p version of [6] says that there exists a class $f_\chi \in P(V)$, which is a product of linear forms, such that

$$P(V)_\chi^G \cong P(V)^{G_\chi} f_\chi^k \in P(V)^{G_\chi}$$

and

$$P(V)^{G_\chi} \cong \bigoplus_{i=0}^{k-1} P(V)^{G_\chi} f_\chi^i.$$

There is the coexact extension sequence

$$\mathbf{F}_p^* \rightarrow P(V)^G \rightarrow P(V)^{G_\chi} \rightarrow P(V)_G^\chi \rightarrow \mathbf{F}_p^*$$

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defining $P(V)_G^X \cong P(V)^{G_X} // P(V)^G$. The algebra $P(V)_G^X$ has the following structure:

$$P(V)_G^X \cong \bigoplus_{i=0}^k f_X^i \cdot F_p,$$

$$f_X^r \cdot f_X^s = \begin{cases} f_X^t & \text{if } \deg f_X^r + \deg f_X^s = \deg f_X^t, \\ 0 & \text{otherwise.} \end{cases}$$

A special case of this situation is when

$$f_X^i = (f_X)^i \in P(V): i = 0, \dots, k-1.$$

In this case we say that $P(V)^{G_X} \geq P(V)^G$ is a *cyclic extension* of $P(V)^G$ of degree k .

If $d = f_X^k$ then we write $P(V) = P(V)^G(\sqrt[k]{d})$ in this case.

Here is a simple example.

EXAMPLE. Let p be an odd prime, s a divisor of $p-1$, and $D(2s) = \mathbf{Z}/s \ltimes \mathbf{Z}/2$ the dihedral group of order $2s$. Embed $D(2s)$ in $GL(2, \mathbf{F}_p)$ by choosing the matrices

$$\begin{bmatrix} \Theta^r & 0 \\ 0 & \Theta^r \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

as generators, where $\Theta \in F_p^*$ is a generator and $r = (p-1)/s$. Then

$$P[x, y]^{D(2s)} = P[xy, x^s + y^s].$$

Let $\Delta: D(2s) \rightarrow F_p^*$ be the determinant character. Then $D(2s)_\Delta = \ker \Delta = \mathbf{Z}/s$. Note

$$P[x, y]_\Delta^{D(2s)} = P[x, y]^{D(2s)} \cdot f_\Delta$$

where $f_\Delta = x^s - y^s$, and

$$P[x, y]^{D(2s)_\Delta} = P[x, y]^{\mathbf{Z}/s} = P[x^s, y^s, xy] / (x^s y^s - (xy)^s).$$

In this case $P(V)^{D(2s)_\Delta}$ is in fact a quadratic extension of $P(V)^{D(2s)}$, namely

$$P(V)^{D(2s)_\Delta} = P(V)^{D(2s)}(\sqrt{d})$$

where $d = (x^s - y^s)^2$.

One of our goals is to prove a converse to Stanley's theorem. We begin, however, with a discussion of an alternate construction procedure for radical extensions. This in turn is based on the observation that, if we place ourselves in the nonmodular characteristic p situation of Stanley's theorem, where

$$P(V)_X^G = P(V)^G f_X,$$

and recall that $P(V)^{G_X}$ is closed under the action of the Steenrod algebra, then

$$\mathcal{P}(f_X) = w f_X$$

for some ungraded class $w = 1 + w_1 + w_2 + \dots \in P(V)^G$, where

$$\mathcal{P} = 1 + P^1 + P^2 + \dots$$

is the total Steenrod operation. In other words, f_X is an eigenvector of the automorphism $\mathcal{P}$. Of course, this follows from Stanley's result that f_X is a product of linear forms. Moreover, the eigenvalue w lies in $P(V)^G$. It is less apparent that

this follows from Stanley's result. Surprisingly enough, the converse of all this holds. To make this precise we introduce a definition.

DEFINITION. $d \in P(V)$ is called *special* iff there exists $w = 1 + w_1 + w_2 + \cdots \in P(V)$ such that $\mathcal{P}(d) = wd$.

THEOREM 1. $d \in P(V)$ is special iff d is a product of two-dimensional classes.

The proof of Theorem 1 is based on the following result of Landweber [5].

THEOREM (LANDWEBER). Let $\mathbf{H}^*$ be an unstable algebra over the Steenrod algebra. Assume $\mathbf{H}^*$ is Noetherian. Then the associated prime ideals of $\mathbf{H}^*$ are invariant under the Steenrod algebra.

There are several different ways to define the associated primes of $\mathbf{H}^*$. For our purposes we take them to be the prime ideals associated to an irredundant primary decomposition of zero in $\mathbf{H}^*$.

Here is a sketch of Landweber's proof. Introduce the total Steenrod operation in the form

$$\mathcal{P}_T: \mathbf{H}^* \rightarrow \mathbf{H}^*[T]; \deg T = -2(p-1)$$

where

$$\mathcal{P}_T(h) = \sum_{i=0}^{\infty} \mathcal{P}^i(h) T^i.$$

Note that $\mathcal{P}_T$ is a homogeneous (degree-preserving) algebra injection. Let

$$0 = \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_r; \mathcal{P}_i = \sqrt{\mathcal{P}_i}, \quad i = 1, \dots, r$$

be an irredundant primary decomposition of zero in $\mathbf{H}^*$. This extends up to the irredundant primary decomposition of zero

$$0 = \mathcal{P}_1[T] \cap \cdots \cap \mathcal{P}_r[T]$$

in $\mathbf{H}^*[T]$. Since $\mathcal{P}_T$ is injective this gives a primary decomposition of zero in $\mathbf{H}^*$

$$0 = \mathcal{P}_T^{-1}(\mathcal{P}_1[T]) \cap \cdots \cap \mathcal{P}_T^{-1}(\mathcal{P}_r[T])$$

with associated primes

$$\mathcal{P}_T^{-1}(\mathcal{P}_i[T]) = \{x \in \mathbf{H}^* \mid P^k(x) \in \mathcal{P}_i \forall k \geq 0\}.$$

Note

$$\mathcal{P}_T^{-1}(\mathcal{P}_i[T]) \subset \mathcal{P}_i \quad \forall i.$$

By uniqueness of associated primes for irredundant primary decompositions (Lasker-Noether) it follows that

$$\mathcal{P}_T^{-1}(\mathcal{P}_i[T]) = \mathcal{P}_i, \quad i = 1, \dots, r.$$

But this says $\mathcal{P}_i$ is invariant, as wished.

As an easy consequence of Landweber's theorem we obtain (N. B. By an *invariant ideal* we mean an ideal invariant under the action of the Steenrod algebra):

COROLLARY 2. Let $\mathbf{R}^*$ be an unstable Noetherian algebra over the Steenrod algebra, and $\mathcal{I}$ an invariant ideal. If

$$\mathcal{I} = \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_r$$

is an irredundant primary decomposition of $\mathcal{J}$ in $\mathbf{R}^*$, then $\mathcal{P}_i = \sqrt{\mathcal{J}_i}$, $i = 1, \dots, r$, are invariant prime ideals of $\mathbf{R}^*$.

PROOF. Pass to $\mathbf{R}^*/\mathcal{J}$, apply Landweber's result, and lift back. ■

PROOF OF THEOREM 1. One direction is clear. Namely, if $d = \prod_{i=1}^s x_i$, $\deg x_1 = \dots = \deg x_s = 2$, then

$$\mathcal{P}d = d \prod_{i=1}^s (1 + x_i^{p-1}),$$

so d is special. Conversely, suppose that

$$\mathcal{P}d = wd: w \in P(V).$$

Then the principal ideal $(d) \subset P(V)$ is invariant. Let

$$d = p_1^{e_1} \cdots p_r^{e_r}$$

be a factorization of d into a product of relatively prime irreducible polynomials. Then

$$(d) = (p_1^{e_1}) \cdots (p_r^{e_r}) = (p_1^{e_1}) \cap \cdots \cap (p_r^{e_r})$$

is an irredundant primary representation of (d) by primary ideals. By the Corollary to Landweber's theorem $(p_1), \dots, (p_r)$ are invariant prime ideals. But by the theorem of Serre and Wilkerson [7] the invariant prime ideals in $P(V)$ are generated by two-dimensional classes. Therefore $\deg p_i = 2$, $i = 1, \dots, r$.

COROLLARY 3. If $d \in P(V)$ is special, and $f \in P(V)$ is such that $f^k = d$, then f is special.

PROOF. By Theorem 1, d is a product of linear forms. $P(V)$ is a unique factorization domain and therefore f is also a product of linear forms and hence special. ■

DEFINITION. Let $f \in P(V)$ and $\mathbf{P}^* \leq P(V)$ be a subalgebra closed under the action of the Steenrod algebra. We say f is $\mathbf{P}^*$ -special $\Leftrightarrow \mathcal{P}f = wf$ with $w \in \mathbf{P}^*$.

We are indebted to Clarence Wilkerson for the following two results, which replace less general results of our own.

PROPOSITION 4. Suppose $\mathbf{P}^* \leq P(V)$ is closed under the action of the Steenrod algebra. Assume $\mathbf{P}^*$ is integrally closed and $P(V)$ is finite over $\mathbf{P}^*$. If $d \in \mathbf{P}^*$ is $P(V)$ -special, then it is $\mathbf{P}^*$ -special.

PROOF. We have

$$\mathcal{P}f = wf: w \in P(V).$$

But $w = \mathcal{P}f/f$ belongs to the field of fractions of $\mathbf{P}^*$. Since $P(V)$ is finite over $\mathbf{P}^*$ the elements of $P(V)$ are integral over $\mathbf{P}^*$. Therefore the homogeneous components of w are integral over $\mathbf{P}^*$ and since $\mathbf{P}^*$ is integrally closed must be contained in it. ■

REMARK. If $\mathbf{P}^* = P(V)^G$, $G \leq GL(V)$, then $\mathbf{P}^*$ satisfies the hypotheses of Proposition 4.

PROPOSITION 5. Suppose $\mathbf{P}^* \leq P(V)$ is closed under the action of the Steenrod algebra and $P(V)$ is finite over $\mathbf{P}^*$. Assume $f \in P(V)$ is $P(V)$ -special. If $f^k \in \mathbf{P}^*$, $(k, p) = 1$, then f is $\mathbf{P}^*$ -special.

PROOF. We need to show $\mathcal{P}^i f/f \in \mathbf{P}^* \forall i$. In fact we will show inductively that $\mathcal{P}^i f/f$ is a polynomial in $\mathcal{P}^1 d/d, \dots, \mathcal{P}^i d/d$. By Proposition 4 the latter are in $\mathbf{P}^*$. We begin with the equations

$$(**) \quad \mathcal{P}_T(f^k)/f^k = \left(\frac{\mathcal{P}_T(f)}{f} \right)^k = 1 + \frac{\mathcal{P}^1 f}{f} T + \dots + f^{p-1} T^N,$$

where $N = -2 \deg(f)(p-1)$ and

$$(\# \#) \quad \frac{\mathcal{P}_T(f^k)}{f^k} = \frac{\mathcal{P}_T(d)}{d} = 1 + \frac{\mathcal{P}^q f}{d} T + \dots + d^{p-1} T^{kN}.$$

Equating gives

$$(*) \quad \left(1 + \frac{\mathcal{P}^1 f}{f} + \dots + f^{p-1} T^N \right)^k = 1 + \frac{\mathcal{P}^1 d}{d} T + \dots + d^{p-1} T^{kN}$$

from which follows

$$k \frac{\mathcal{P}^1 f}{f} = \frac{\mathcal{P}^1 d}{d},$$

and since k is a unit mod p ,

$$\frac{\mathcal{P}^1 f}{f} = k^{-1} \frac{\mathcal{P}^1 d}{d}.$$

This allows us to start an induction. From $(*)$ we get

$$k \frac{\mathcal{P}^j f}{f} + G_j \left(\frac{\mathcal{P}^1 f}{f}, \dots, \frac{\mathcal{P}^{j-1} f}{f} \right) = F_j \left(\frac{\mathcal{P}^1 d}{d}, \dots, \frac{\mathcal{P}^j d}{d} \right).$$

By induction $G_j(\)$ and $F_j(\) \in \mathbf{P}^*$, and since k is a unit mod p the result follows. ■

As a consequence of this discussion we obtain the following procedure for constructing cyclic extensions.

Suppose $G \leq GL(V)$, and $d \in P(V)^G$ is special. If d has a k th root f in $P(V)$, where k is a divisor of $p-1$. Then

$$\mathcal{P}f = uf, \quad u \in P(V)^G.$$

Therefore $\mathbf{R}^*$ is the subalgebra of $P(V)$ generated by $P(V)^G$ and f , $\mathbf{R}^*$ is closed under the action of the Steenrod algebra. Therefore $\mathbf{R}^*$ is a radical extension of $P(V)^G$. Note that the degree of the extension is k' where k' is the smallest divisor of k such that $f^{k'} \in P(V)^G$. Finally, note that to check whether d has a k th root in $P(V)$ is an easy task, because by Theorem 1 d is a product of linear forms. If we write

$$d = v_1^{e_1} \cdots v_r^{e_r},$$

with $v_1, \dots, v_r \in V$ relatively prime linear forms, then d has a k th root iff $e_i \equiv 0 \pmod k$, $i = 1, \dots, r$.

We next prove the converse of Stanley's theorem in the nonmodular characteristic p setting. We require some definitions.

DEFINITION. Let $\mathbf{T}^*$ be a graded connected totally finite algebra over $\mathbf{F}_p$. We say $\mathbf{T}^*$ has *maximally nonzero multiplication* iff

$$\dim T^j \leq 1 \quad \text{and} \quad \mu: T^i \otimes T^j \rightarrow T^{i+j} \text{ is surjective.}$$

Note that this is equivalent to saying that $\mathbf{T}^*$ has a basis $1 = f_0, \dots, f_k$, $k = \dim \text{Tot}(\mathbf{T}^*)$, where

$$f_r \cdot f_s = \begin{cases} f_t & \text{if } \deg(f_r) + \deg(f_s) = \deg(f_t), \\ 0 & \text{otherwise.} \end{cases}$$

DEFINITION. Let $\mathbf{P}^*$ be an unstable polynomial algebra over the Steenrod algebra. A *cyclic extension of $\mathbf{P}^*$ of degree k* is a coexact sequence

$$\mathbf{F}_p \rightarrow \mathbf{P}^* \rightarrow \mathbf{R}^* \rightarrow \mathbf{T}^* \rightarrow \mathbf{F}_p,$$

where $\mathbf{R}^*$ is a free $\mathbf{P}^*$ of rank k , and $\mathbf{T}^*$ is a faithful $\mathbf{Z}/k$ algebra with maximally nonzero multiplication. (N.B. If H is a group, then an H -algebra is an algebra over the Hopf algebra $\mathbf{F}_p[H]$. It is faithful if the action of H is faithful. Note that $P[f]/(f^k)$, where k is a divisor of $p-1$, is a $\mathbf{Z}/k$ algebra in a natural way, where the action of a generator γ of $\mathbf{Z}/k$ is given by:

$$\gamma f = \Theta^m f: m = (p-1)/m$$

and θ is a primitive $(p-1)$ st root of unity. This is the situation when $\mathbf{R}^* = \mathbf{P}^*(\sqrt[k]{d})$.)

THEOREM 6. Let $\mathbf{R}^* \geq \mathbf{P}^*$ be a cyclic extension of degree k , $k \mid p-1$, where $\mathbf{R}^*$ and $\mathbf{P}^*$ are generated by classes of degrees relatively prime to p , $\mathbf{P}^*$ is a polynomial algebra and $\mathbf{R}^*$ is integrally closed. Then there exists $G \leq GL(V)$ generated by pseudoreflections of order relatively prime to p and a linear character $\chi: G \rightarrow \mathbf{F}_p^*$ of degree k such that

$$\begin{array}{ccc} \mathbf{R}^* & \simeq & P(V)^{G_\chi} \\ \vee & & \vee \\ \mathbf{P}^* & \simeq & P(V)^G. \end{array}$$

PROOF. By hypothesis $\mathbf{R}^*$ is integrally closed and therefore we may apply [1] to embed $\mathbf{R}^* \hookrightarrow P(V)$, for some V , with $P(V)$ the algebraic closure of $\mathbf{R}^*$ in the category of unstable domains over the Steenrod algebra. Since $\mathbf{P}^*$ and $\mathbf{R}^*$ have the same transcendence degree, the composite embedding $\mathbf{P}^* \hookrightarrow \mathbf{R}^* \hookrightarrow P(V)$ is the embedding of $\mathbf{P}^*$ into its algebraic closure also. Therefore by [1] $\mathbf{P}^* = P(V)^G$ for some $G \leq GL(V)$ generated by pseudoreflections of order relatively prime to p . Let

$$\mathbf{R}^* = \bigoplus_{i=0}^{k-1} \mathbf{P}^* \cdot f_i$$

and identify $\{1 = f_0, \dots, f_{k-1}\}$ with a basis for $\mathbf{T}^*$. This defines an action of $\mathbf{Z}/k$ on $\mathbf{R}^*$. Let $\gamma \in \mathbf{Z}/k$ be a generator. Suppose

$$r = h_0 + h_1 f_1 + \dots + h_{k-1} f_{k-1}$$

and $\gamma r = r$. Then

$$h_0 + h_1 f_1 + \cdots + h_{k-1} f_{k-1} = h_0 + h_1 \gamma f_1 + \cdots + h_{k-1} \gamma f_{k-1}.$$

But $\{1 = f_0, f_1, \dots, f_{k-1}\}$ is a basis for $\mathbf{R}^*$ over $\mathbf{P}^*$ and hence

$$f_i = \gamma f_i: i = 1, \dots, k-1$$

and since $\mathbf{T}^*$ is faithful this means

$$h_1 = h_2 = \cdots = h_{k-1} = 0,$$

so $r \in \mathbf{P}^*$. Thus $\mathbf{R}^*$ is Galois over $\mathbf{P}^*$ with Galois group $\mathbf{Z}/k$. Let $F(\)$ be the field of fractions functor. Then we have

$$\begin{array}{c} F(\mathbf{P}^*) < F(\mathbf{R}^*) < F(P(V)) \\ \hline \mathbf{Z}/k \\ \hline G \end{array}$$

By the fundamental theorem of Galois theory $F(\mathbf{R}^*) \leq P(V)$ is Galois with Galois group $H \triangleleft G$ and $G/H \cong \mathbf{Z}/k$. If

$$\chi: G \rightarrow G/H \cong \mathbf{Z}/k < \mathbf{F}_p^*$$

is the composite linear character, we have $H = G_\chi$. Since everything in sight is integrally closed we may lift to obtain

$$\begin{array}{c} F(\mathbf{P}^*) < F(\mathbf{R}^*) < F(P(V)) \\ \hline \mathbf{Z}/k \quad G_\chi \\ \hline G \end{array}$$

as required. ■

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