EXTERNAL EQUIVALENCE CLASSES IN DECOMPOSITIONS OF SPACES

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ABSTRACT. Let X be an indecomposable continuum in the plane. Krasinkiewicz has shown that the union of the external composants of X is a first category F_σ set in X . We give extensions of this theorem to quite general decompositions of a space X embedded in a locally connected ambient space Y .

1. Introduction. Consider the Cantor ternary set X in the real line E_1 . Let X' denote the set of points of X which are not endpoints of any interval in $E_1 \setminus X$. If L is an arc in E_1 which meets X' , then $L \cap X$ contains an open and, hence, a second category set in X .

Mazurkiewicz [5] proved that if X is an indecomposable continuum in the plane, then the union of the accessible composants of X is a first category F_σ -set in X . Krasinkiewicz [3] has given a far reaching generalization of Mazurkiewicz's theorem.

In this paper we will prove two theorems which will simultaneously generalize the above.

All spaces we consider are separable and metric. A *continuum* is a compact, connected metric space. If (X, d) is a bounded metric space we let $C(X)$ denote the *hyperspace* of *subcontinua* of X with the Hausdorff metric.

If $A \subset X$ we let $\text{Cl}_X(A)$, $\text{Bd}_X(A)$, and $\text{Int}_X(A)$ denote the closure, boundary, and interior, respectively, of A in X .

Let X be a subspace of a space Y and let R be an equivalence relation defined on X . For $x \in X$ we let $R(x)$ denote the R -equivalence class of x . For $A \subset X$ let $R(A) = \bigcup_{x \in A} R(x)$. An R -equivalence class $R(x)$ is said to be *external in the sense of Krasinkiewicz* if there exists a continuum L in Y such that $L \not\subset \text{Cl}(X)$ and $R(x) \subset R(L \cap X) \neq X$. We let E_{Kras} denote the union of the external (in the sense of Krasinkiewicz) R -equivalence classes of X .

An R -equivalence class $R(x)$ is said to be *external in the sense of category* if there exists a continuum L in Y such that $L \not\subset \text{Cl}(X)$ and $R(x) \subset R(L \cap X)$ is of first category in X . We let E_{cat} denote the union of the external (in the sense of category) R -equivalence classes of X .

A continuum X is *indecomposable* if it is not the union of two proper subcontinua. If $x \in X$, the *composant* of x in X is the union of all proper subcontinua of X which contain x . It is well known that the composants of an indecomposable continuum X are dense, pairwise disjoint, first category, F_σ -sets in X . Also, the equivalence relation R on X whose equivalence classes are the composants of X has the property that $R = \bigcup_{i=1}^{\infty} R_i$ where each R_i is closed in $X \times X$.

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Our first proposition shows that for the very important case of indecomposable continua in the 2-sphere the two notions of externality coincide.

PROPOSITION 1. *If X is a compact space contained in a space Y and $R = \bigcup_{i=1}^{\infty} R_i$ is an equivalence relation on X such that each R_i is closed in $X \times X$ and each $R(x)$ is dense in X , then $Ekras = Ecat$.*

PROOF. Since X is of second category it is clear that $Ecat \subset Ekras$. If $R(z) \subset Ekras$ let K be a continuum in Y such that $K \not\subset Cl_Y(X)$ and $R(z) \subset R(K \cap X) \neq X$. For each positive integer i let

$$L_i = \{y \in R(K \cap X) \mid (x, y) \in R_i \text{ for some } x \in K\}.$$

Then L_i is closed in X and $R(K \cap X) = \bigcup_{i=1}^{\infty} L_i$. Since $R(K \cap X) \neq X$ and each $R(x)$ is dense in X it follows that L_i is nowhere dense in X . Hence, $R(K \cap X)$ is of first category in X .

COROLLARY 2. *Let X be an indecomposable continuum contained in a space Y and let R be the decomposition of X into composants. Then $Ecat = Ekras$, i.e. the two definitions of external composants coincide.*

Krasinkiewicz [3] proved the following remarkable extension of Mazurkiewicz's theorem:

THEOREM (KRASINKIEWICZ). *If X is an indecomposable continuum in the 2-sphere S^2 and R is the decomposition of X into composants, then $Ekras$ is a first category F_{σ} -set in X .*

In [2] the authors extended Krasinkiewicz's theorem in several ways.

THEOREM (DEBSKI AND TYMCHATYN). *Let $X \subset S^2$ be such that each nonempty open subset of X is of second category. Let R be an equivalence relation on X such that each $R(x)$ is continuum connected, of first category in X , and dense in X . Then $Ekras$ is a first category F_{σ} -set in X .*

It is the purpose of this paper to prove analogues of some of the theorems in [2] concerning $Ekras$ in S^2 for $Ecat$ in general ambient spaces.

2. Main results.

THEOREM 3. *Let X be a subspace of a locally connected separable, metric space Y . Let R be an equivalence relation on X such that each $R(x)$ is continuum connected. Suppose $X' \subset X$ such that $X \setminus X'$ is of first category in X and for each ordered pair (U, V) of disjoint open sets in Y with $U \cap X \neq \emptyset \neq V \cap X$ and for each $x \in X' \cap V$, $Y \setminus U$ is separated by $R(x)$ between two points of $V \cap X$. Then $Ecat$ is a first category set in X .*

PROOF. Clearly, we may suppose X is of second category or there is nothing to prove. Let $\mathcal{V} = \{V_1, V_2, \dots\}$ be a basis of open connected sets for the topology of Y . Let $\mathcal{U} = \{U_1, U_2, \dots\}$ be the elements of $\mathcal{V}$ which meet X . If $Cl_Y(V_i) \cap (Cl_Y(X \cup U_n)) = \emptyset$ let

$$D_{n,i} = \bigcup \{C \cap X \mid C \in \mathcal{C}(Y \setminus U_n), C \cap Cl_Y(V_i) \neq \emptyset \\ \text{and } R(C \cap X) \text{ is of first category in } X\}.$$

Otherwise, let $D_{n,i} = \emptyset$.

Clearly, $D_{n,i} \subset \text{Ecat}$. Now, let $R(x)$ be an external R -equivalence class of X . Let L be a continuum in Y which meets both $R(x)$ and $Y \setminus \text{Cl}_Y(X)$ and such that $R(L \cap X)$ is of first category in X . Let K be a continuum in $R(x)$ from x to L . Then $R((K \cup L) \cap X) = R(L \cap X)$ so there exists $y \in X \setminus R(L \cap X)$. Let U_n be a basic neighborhood of y such that $\text{Cl}_Y(U_n) \cap (L \cup K) = \emptyset$. Let $z \in L \setminus \text{Cl}_Y(X)$ and let V_i be a basic neighborhood of z such that $\text{Cl}_Y(V_i) \cap \text{Cl}_Y(U_n \cup X) = \emptyset$. Then $x \in D_{n,i}$. So $\bigcup D_{n,i} = \text{Ecat}$.

Let n and i be fixed integers. To prove Ecat is of first category in X it suffices to prove that the interior in X of $\text{Cl}_X(D_{n,i})$ is of first category in X . Just suppose m is a positive integer such that $U_m \cap (V_i \cup U_n) = \emptyset$ and $U_m \cap X \subset \text{Cl}_X(D_{n,i})$. Let $L_1, L_2, \dots$ be continua in $Y \setminus U_n$ such that for each j $L_j \cap \text{Cl}_Y(V_i) \neq \emptyset$, $R(L_j \cap X)$ is of first category in X , and $U_m \cap X \subset \text{Cl}_X(\bigcup_{j=1}^{\infty} L_j \cap X)$.

Let $x \in X' \cap U_m$. Then $R(x)$ separates $Y \setminus U_n$ between two points of $X \cap U_m$. Since V_i is connected, $R(x)$ separates $Y \setminus U_n$ between V_i and a point of $X \cap U_m$. Since Y is hereditarily normal, a closed subset of $R(x)$ separates $Y \setminus U_n$ between V_i and a point of $X \cap U_m$. Since $\bigcup_{j=1}^{\infty} L_j \cap X$ is dense in $X \cap U_m$ and each L_j meets $\text{Cl}_Y(V_i)$ it follows that $R(x)$ meets some L_j .

Hence $X' \cap U_m \subset \bigcup_{j=1}^{\infty} R(L_j \cap X)$. It follows that U_m is of first category in X since $R(L_j \cap X)$ and $X \setminus X'$ are of first category in X . Since the interior in X of $\text{Cl}_X(D_{n,i})$ is contained in the union of the countable family of sets $\{U_j \cap X | j = 1, 2, \dots\}$ it follows that the interior in X of $\text{Cl}_X(D_{n,i})$ is of first category in X .

We give an example to show that the condition in Theorem 3 that $R(x)$ is continuum connected is essential.

EXAMPLE 4. Let K_2 be Knaster's simplest indecomposable continuum,

$$K_2 = \varprojlim (I_i, f_i^j),$$

where each $I_i = [0, 1]$ and $f_i^{i+1} = f$ is the mapping $f: [0, 1] \rightarrow [0, 1]$ given by

$$f(x) = \begin{cases} 2x & \text{if } x \in [0, \frac{1}{2}], \\ 2 - 2x & \text{if } x \in [\frac{1}{2}, 1]. \end{cases}$$

Let K_2 have its usual embedding in the plane (see [4, p. 205]). Let $\underline{0} = (0, 0, 0, \dots)$ be the endpoint of K_2 . Let F be the composant of K_2 which contains $\underline{0}$. Then each point of F is accessible. Let

$$F^1 = F \setminus \{(x, x/2, x/4, \dots) | 0 \leq x < \frac{1}{2}\}.$$

Since $F \setminus F^1$ has c points and K_2 has c composants let $g: F \setminus F^1 \rightarrow \{C | C \text{ is a composant of } K_2\}$ be a one-to-one onto correspondence. Define an equivalence relation R on K_2 by letting $R(x) = \{x\} \cup g(x)$ for each $x \in F \setminus F^1$ such that $g(x) \neq F$. If $g(x) = F$ let $R(x) = (F \setminus F^1) \cup \{x\}$. Each $R(x)$ is connected (but not continuum connected) since composants of K_2 and $F \setminus F^1$ are dense connected sets in K_2 . Krasinkiewicz showed that each composant of K_2 different from F separates S^2 locally in the required way. Each R -equivalence class of K_2 is accessible in the plane.

THEOREM 5. Let X be a subspace of a compact, locally connected, metric space Y . Suppose R is an equivalence relation on X such that each $R(x)$ is continuum connected, dense in X , and of first category in X . Suppose $X' \subset X$ such that

$X \setminus X'$ is of first category in X and for each pair (U, V) of open sets in Y such that $U \cap X \neq \emptyset \neq V \cap X$ and for each $x \in X'$, $R(x)$ separates $Y \setminus U$ between two points of $X \cap V$. Then $Ecat$ is a first category F_σ -set in X .

PROOF. Let $\mathcal{V}, \mathcal{U}$, and $D_{n,i}$ be as in Theorem 3. Then $Ecat = \bigcup D_{n,i}$ as in Theorem 3. To prove the theorem it suffices to prove that each $D_{n,i}$ is closed in X .

Let n and i be fixed positive integers and let $\{x_j\}_{j=1}^\infty$ be a sequence in $D_{n,i}$ which converges to $x \in X$. For each positive integer j let L_j be a continuum in $Y \setminus U_n$ from x_j to $Cl_Y(V_i)$ and such that $R(L_j \cap X)$ is of first category in X . By passing to a subsequence if necessary we may suppose $\{L_j\}_{j=1}^\infty$ converges to a continuum L in the compact metric space $C(Y \setminus U_n)$. Then $x \in L$ and $L \cap Cl_Y(V_i) \neq \emptyset$.

If $R(L \cap X)$ is of first category in X we are done. Suppose, therefore, that $R(L \cap X)$ is of second category in X . By Theorem 3 $Ecat$ is of first category in X . Hence, $R(L \cap X) \setminus ((X \setminus X') \cup Ecat) \neq \emptyset$. Let $y \in R(L \cap X) \setminus ((X \setminus X') \cup Ecat)$. Since $R(y)$ is dense in X we may suppose without loss of generality that $y \in U_n$. Let $\{U_{n,j}\}_{j=1}^\infty$ be a basis of neighborhoods of y chosen from $\mathcal{U}$ and such that $U_{n,j} \subset U_n$ for each j .

For each $z \in R(L \cap X) \setminus R(y)$ and each j let $A_{z,j}$ denote the union of the continua in $R(z) \setminus U_{n,j}$ which meet L . Let $Q_j = \bigcup \{A_{z,j} | z \in R(L \cap X) \setminus R(y)\}$. Since each $R(z)$ is continuum connected, $R(L \cap X) = \bigcup_{j=1}^\infty Q_j \cup R(y)$. Since $R(L \cap X)$ is of second category in X and $R(y)$ is of first category in X there exists an integer r such that Q_r is of second category in X . Hence, there exists an integer m such that Q_r is dense in $U_m \cap X$ and $U_m \subset Cl_Y(U_m) \subset Y \setminus Cl_Y(U_{n,r})$.

Since $y \in X'$, $R(y)$ separates $Y \setminus U_{n,r}$ between two points of $U_m \cap X$. Since Y is hereditarily normal, a closed subset A of $R(y)$ separates $Y \setminus U_{n,r}$ between two points of $U_m \cap X$. Since Q_r is dense in $U_m \cap X$, A separates $Y \setminus U_{n,r}$ between two points a and b of Q_r . By the definition of $A_{a,r}$ and $A_{b,r}$ there exist continua in $R(a)$ (resp. $R(b)$) from a to L (resp. from b to L). These continua miss A . Hence, A separates two points a' and b' of L in $Y \setminus U_{n,r}$. Since the sequence $\{L_j\}_{j=1}^\infty$ converges to L there is a positive integer p such that A separates two points of L_p in $Y \setminus U_{n,r}$. Hence, $A \cap L_p \subset R(y) \cap L_p \neq \emptyset$. This is a contradiction since $R(y) \not\subset Ecat$. The proof is complete.

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