FREE PRODUCTS OF LATTICE-ORDERED GROUPS

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Dedicated to Sydney Berlanny—in memoriam

ABSTRACT. We give a highly homogeneous representation for the free product of two nontrivial countable lattice-ordered groups and obtain, as a consequence of the method, that the free product of nontrivial lattice-ordered groups is directly indecomposable and has trivial center.

1. Introduction. The free lattice-ordered group on a countably infinite set of generators was shown to be isomorphic to a doubly transitive sublattice subgroup of the lattice-ordered group of all order-preserving permutations of the rational line (see [1 or 2, Theorem 6.7, and 9]). The proof is easily extended to give a faithful doubly transitive representation for the free lattice-ordered group on any infinite set of generators. More recently, Kopytov [6] and, independently, McCleary [9] proved by a far more subtle technique that the free lattice-ordered group on at least two generators possesses a faithful doubly transitive representation (on the rational line if the number of generators is finite or countable). The ideas were further extended in [3] to prove:

1. *If G is any lattice-ordered group and F is a free lattice-ordered group on an infinite set of generators of cardinality at least $|G|$, then the free product (in the category of lattice-ordered groups) of G and F has a faithful doubly transitive representation on some ordered field of cardinality $|F|$.*

2. *If G is any countable lattice-ordered group and F is a free lattice-ordered group on a finite number (> 1) of generators, then the free product (in the category of lattice-ordered groups) of G and F has a faithful doubly transitive representation on the rational line.*

Although the proof of 2 is the more delicate, the purpose of this note is to extend it. (In contrast, I have been unable to extend the proof of 1.) Specifically,

THEOREM. *The free product (in the category of lattice-ordered groups) of any two nontrivial countable lattice-ordered groups has a faithful doubly transitive representation on the rational line.*

Now doubly transitive sublattice subgroups of the lattice-ordered group of all order-preserving permutations of the rational line have trivial center [2, Lemma 1.6.8] and are directly indecomposable (in the category of lattice-ordered groups) [11, Lemma 2]—indeed, in the category of groups [4]. Hence

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COROLLARY 1. *The free product (in the category of lattice-ordered groups) of any two nontrivial countable lattice-ordered groups has trivial center and is directly indecomposable (in this category).*

The proof of the theorem, although it does not yield a faithful doubly transitive representation of the free product of arbitrary nontrivial lattice-ordered groups, is adaptable to show:

COROLLARY 1'. *The free product (in the category of lattice-ordered groups) of any family of nontrivial lattice-ordered groups has trivial center and is directly indecomposable (in this category or the category of groups).*

CAUTION. Corollary 1' does not follow immediately from Corollary 1. Indeed, we would like to know if:

Whenever G_0 and H_0 are (finitely generated) sublattice subgroups of the lattice-ordered groups G and H respectively, is the sublattice subgroup of the free product (in the category of lattice-ordered groups) of G and H generated by $G_0 \cup H_0$ isomorphic to the free product (in this category) of G_0 and H_0 ?

Powell and Tsinakis [12] proved the direct indecomposability of free products of nontrivial Abelian lattice-ordered groups (in the category of Abelian lattice-ordered groups) and of nontrivial finitely generated lattice-ordered groups belonging to any proper subvariety of lattice-ordered groups (the free product being taken in the variety) [13, Corollary 8.7], but leave open the general problem for free products in the variety of all lattice-ordered groups [13, Problem 10.9]. Corollary 1' therefore answers their question for the variety of all lattice-ordered groups.

In any doubly transitive sublattice subgroup of the lattice-ordered group of all order-preserving permutations of a totally ordered set, there is a countable (infinite) pairwise disjoint set. Thus

COROLLARY 2. *The free product (in the category of lattice-ordered groups) of any two nontrivial countable lattice-ordered groups contains a countable set of pairwise disjoint elements.*

This partially answers [13, Problem 10.7].

Any nontrivial lattice-ordered group which has a faithful doubly transitive representation generates the variety of all lattice-ordered groups (see [2, p. 247]). Hence

COROLLARY 3. *The free product (in the category of lattice-ordered groups) of any two nontrivial countable lattice-ordered groups generates the variety of all lattice-ordered groups.*

This improves Theorem 2.1 and Corollary 2.1.1 of [7]. Again the proof of Corollary 1' extends the result to an arbitrary family of nontrivial lattice-ordered groups so that the word "countable" can be omitted from the statement of Corollary 3.

CAUTION. If G and H are arbitrary lattice-ordered groups, then

$$[[g \vee e, h^{-1}(g^{-1} \vee e)h], [h^{-1}(g \vee e)h, g^{-1} \vee e]] = e$$

holds in the free product (in the category of lattice-ordered groups) of G and H for any $g \in G$ and $h \in H$, where $[a, b] = a^{-1}b^{-1}ab$; so if G is not totally ordered and $g \in G$ with $g \not\geq e$ and $g \not\leq e$, the subgroup of this free product generated by $G \cup H$ is not the group free product of G and H (see [5]).

2. Background. We recall some definitions and facts from [2].

A group that is also a lattice which satisfies the identities $x(y \vee z)t = xyt \vee xzt$ and $x(y \wedge z)t = xyt \wedge xzt$ is said to be a *lattice-ordered group*. Lattice-ordered groups are torsion-free ($x^m = e \rightarrow x = e$ if m is a positive integer and e denotes the identity element) distributive lattices ($x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$) (see [2, Lemmas 1.11.1 and 1.11.2]). Thus every lattice-ordered group expression $w(\mathbf{x})$ can be rewritten in the form $\bigvee_i \bigwedge_j w_{ij}$, where i and j range over finite index sets and w_{ij} are just group expressions, say $w_{ij} = x_{ij0} \cdots x_{ijk}$ with $x_{ij0} = e$ and each x_{ijn} a generator, the inverse of a generator, or e . The set $\{x_{ij0} \cdots x_{ijn} : 0 \leq n \leq k, i \in I, j \in J\}$ is called the set of *initial subwords* of w .

Let $\mathbf{R}$ denote the real line and $\Omega \subseteq \mathbf{R}$ with the inherited order. We write $A(\Omega)$ for the lattice-ordered group of all order-preserving permutations of $(\Omega, \leq)$ where $f \leq g$ if and only if $\alpha f \leq \alpha g$ for all $\alpha \in \Omega$. By [2, Lemmas 2.5.1 and 2.5.2 and Holland's Theorem (Appendix I)], any countable lattice-ordered group can be embedded (as a lattice-ordered group) in $A(\mathbf{Q})$, where $\mathbf{Q}$ denotes the rational line. A sublattice subgroup G of $A(\Omega)$ is said to be *doubly transitive* if whenever $\alpha, \beta, \gamma, \delta \in \Omega$ with $\alpha < \beta$ and $\gamma < \delta$, there exists $g \in G$ such that $\alpha g = \gamma$ and $\beta g = \delta$. (Such sublattice subgroups are m -transitive for every finite m —see [2, Lemma 1.10.1].) A lattice-ordered group isomorphic to such a sublattice subgroup is said to have a *faithful doubly transitive representation on Ω* . Finally, if $f \in A(\Omega)$, then $\text{supp}(f) = \{\alpha \in \Omega : \alpha f \neq \alpha\}$ is called the *support* of f . A doubly transitive sublattice subgroup of $A(\Omega)$ in which no nonidentity element has support bounded both above and below in Ω is called *pathological*. Pathological lattice-ordered groups are never completely distributive [8]. It will follow from the proof of the theorem that the representation is pathological and, consequently, free products of nontrivial lattice-ordered groups are never completely distributive.

For any terms and background that I have unwittingly omitted, see [2].

3. Proof of the Theorem. Let $G * H$ denote the free product of G and H in the category of lattice-ordered groups. Since G and H are countable and nontrivial, so is $G * H$. It follows (see §2) that $G * H$ can be embedded in $A(\Delta)$ for any nondegenerate open interval Δ of the rational line $\mathbf{Q}$. Let $w_0, w_1, w_2, \dots$ be an enumeration of the nonidentity elements of $G * H$. Let

$$\begin{aligned} \Delta_n &= (2n - \tfrac{1}{9}, 2n + \tfrac{10}{9}), & \Lambda_n &= (2n - \tfrac{1}{3}, 2n + \tfrac{4}{3}), \\ A_{n,0} &= (2n + \tfrac{11}{9}, 2n + \tfrac{13}{9}), & A_{n,1} &= (2n + \tfrac{14}{9}, 2n + \tfrac{16}{9}) \end{aligned}$$

and

$$B_n = (2n + \tfrac{25}{18}, 2n + \tfrac{29}{18}), \quad n \in \{0, 1, 2, \dots\} = \omega.$$

Hence

$$\Delta_n \subseteq \Lambda_n, \quad \Lambda_n < B_n < \Lambda_{n+1}, \quad \Delta_n < A_{n,0} < A_{n,1} < \Delta_{n+1}$$

with $A_{n,0} \cap \Lambda_n \neq \emptyset \neq A_{n,1} \cap \Lambda_{n+1}$, and $A_{n,i} \cap B_n \neq \emptyset$ ($i = 0, 1$) for all $n \in \omega$.

Embed $G * H$ in $A(\Delta_n)$ by ϕ_n , say ($n \in \omega$). Clearly we may assume that for some $\alpha_n \in \Delta_n$ with $2n \leq \alpha_n < 2n + 1$,

(1) $\alpha_n w_n \neq \alpha_n$, $2n = \min\{\alpha_n(u\phi_n) : u \text{ initial subword of } w_n\}$ and $2n + 1 = \max\{\alpha_n(u\phi_n) : u \text{ initial subword of } w_n\}$.

For each $n \in \omega$, define $\theta_n: H \rightarrow A(\Lambda_n)$ as follows:

(a) If every element of $H\phi_n$ fixes $2n$ and $2n+1$, let $g_{n,0}, g_{n,1} \in G$ be such that $(2n)(g_{n,0}\phi_n) \leq 2n$ and $(2n+1)(g_{n,1}\phi_n) > (2n+1)$. (Since $2n = \alpha_n(u_0\phi_n)$ and $2n+1 = \alpha_n(u_1\phi_n)$ for some initial subwords u_0, u_1 of w_n , such elements of G must exist.) Let $\theta_{n,i}$ ($i = 0, 1$) be embeddings of H in $A((2n - \frac{1}{3}, 2n))$ and $A((2n+1, 2n + \frac{4}{3}))$ respectively such that

$$(2n)(g_{n,0}\phi_n)(h_{n,0}\theta_{n,0}) = 2n - \frac{2}{9} \quad \text{and} \quad (2n+1)(g_{n,1}\phi_n)(h_{n,1}\theta_{n,1}) = 2n + \frac{11}{9}$$

for some $h_{n,0}, h_{n,1} \in H$. If $h \in H$, let

$$\lambda(h\theta_n) = \begin{cases} \lambda(h\phi_n) & \text{if } \lambda \in [2n, 2n+1], \\ \lambda(h\theta_{n,0}) & \text{if } \lambda \in (2n - \frac{1}{3}, 2n), \\ \lambda(h\theta_{n,1}) & \text{if } \lambda \in (2n+1, 2n + \frac{4}{3}). \end{cases}$$

(b) If every element of $H\phi_n$ fixes $2n$ but for some element $h_{n,1} \in H$, $2n+1 < (2n+1)(h_{n,1}\phi_n)$, let $a_n \in A(\Lambda_n)$ be the identity on $(2n - \frac{1}{3}, 2n+1]$ with $2n + \frac{11}{9} = (2n+1)(h_{n,1}\phi_n)a_n$. For $h \in H$, let

$$\lambda(h\theta_n) = \begin{cases} \lambda(a_n^{-1}(h\phi_n)a_n) & \text{if } \lambda \in [2n, 2n + \frac{4}{3}], \\ \lambda(h\theta_{n,0}) & \text{if } \lambda \in (2n - \frac{1}{3}, 2n), \end{cases}$$

where $\theta_{n,0}$ is defined as in (a).

(c) If every element of $H\phi_n$ fixes $2n+1$ but for some element $h_{n,0} \in H$, $(2n)(h_{n,0}\phi_n) < 2n$, proceed dually to (b).

(d) If for some $h_{n,0}, h_{n,1} \in H$, $(2n)(h_{n,0}\phi_n) < 2n$ and $2n+1 < (2n+1)(h_{n,1}\phi_n)$, let $a_n \in A(\Lambda_n)$ be the identity on $[2n, 2n+1]$ with $(2n)(h_{n,0}\phi_n)a_n = 2n - \frac{2}{9}$ and $2n + \frac{11}{9} = (2n+1)(h_{n,1}\phi_n)a_n$. Let $h\theta_n = a_n^{-1}(h\phi_n)a_n$ for all $h \in H$.

Note that in all cases θ_n is an embedding and that whereas $\text{supp}(g\phi_n) \subseteq \Delta_n$ for all $g \in G$, $\text{supp}(h\theta_n)$ overflows into $\Lambda_n \setminus \Delta_n$ at both ends for appropriate elements of $H\theta_n$, namely $h_{n,0}\theta_n$ and $h_{n,1}\theta_n$.

Let $\phi_{n,i}: G \rightarrow A(A_{n,i})$ ($i = 0, 1$) be embeddings such that

$$(2n + \frac{11}{9})(h_{n,1}\theta_n)(\bar{g}_{n,0}\phi_{n,0}) = 2n + \frac{25}{18}$$

and

$$(2n + \frac{16}{9})(h_{n+1,0}\theta_{n+1})(\bar{g}_{n,1}\phi_{n,1}) = 2n + \frac{29}{18}$$

for some $\bar{g}_{n,0}, \bar{g}_{n,1} \in G$, and let $\bar{\phi}_n: G \rightarrow A((2n - \frac{1}{3}, 2n + \frac{5}{3}))$ be given by

$$\beta(g\bar{\phi}_n) = \begin{cases} \beta(g\phi_n) & \text{if } \beta \in \Delta_n, \\ \beta(g\phi_{n,i}) & \text{if } \beta \in A_{n,i} \text{ } (i = 0, 1), \\ \beta & \text{otherwise.} \end{cases}$$

Then $\bar{\phi}_n$ is an embedding of G into $A((2n - \frac{1}{3}, 2n + \frac{5}{3}))$.

Let $\hat{\theta}_n: H \rightarrow A(B_n)$ be any embedding such that

$$(2n + \frac{25}{18})(\bar{g}_{n,0}\phi_{n,0})(h_n\hat{\theta}_n) = (2n + \frac{29}{18})(\bar{g}_{n,1}\phi_{n,1})$$

for some $h_n \in H$, and define $\bar{\theta}_n: H \rightarrow A((2n - \frac{1}{3}, 2n + \frac{5}{3}))$ by

$$\beta(h\bar{\theta}_n) = \begin{cases} \beta(h\hat{\theta}_n) & \text{if } \beta \in B_n, \\ \beta(h\theta_n) & \text{if } \beta \in \Lambda_n, \\ \beta & \text{otherwise.} \end{cases}$$

Then $\bar{\theta}_n$ is an embedding.

Let $\bar{\alpha}_0 = \inf\{-\frac{2}{9}(h\bar{\theta}_0) : h \in H\} \in \mathbf{R}$; so $\bar{\alpha}_0 \geq -\frac{1}{3}$. Let $\hat{\phi}_0$ be an embedding of G into $A(\mathbf{Q})$ such that if $\Gamma = \{-\frac{2}{9}(h_{0,0}\bar{\theta}_0)(g\hat{\phi}_0) : g \in G\}$, then $\sup \Gamma = -\frac{2}{9}$ and $\inf \Gamma = \bar{\alpha}_0$.

We now define maps of G and H into $A(\Omega_0)$, where $\Omega_0 = \{\alpha \in \mathbf{Q} : \alpha > \bar{\alpha}_0\}$. For $g \in G$, let

$$\beta(g\phi) = \begin{cases} \beta(g\bar{\phi}_n) & \text{if } \beta \in (2n - \frac{1}{3}, 2n + \frac{5}{3}) \text{ for some } n \in \omega \setminus \{0\}, \\ \beta(g\bar{\phi}_0) & \text{if } \beta \in (-\frac{2}{9}, \frac{5}{3}), \\ \beta(g\hat{\phi}_0) & \text{if } \beta \in (\bar{\alpha}_0, -\frac{2}{9}). \end{cases}$$

Then ϕ is a well-defined homomorphism since $\hat{\phi}_0$ and each $\bar{\phi}_n$ is. For $h \in H$, let

$$\beta(h\theta) = \begin{cases} \beta(h\bar{\theta}_0) & \text{if } \beta \in (2n - \frac{1}{3}, 2n + \frac{5}{3}) \text{ for some } n \in \omega \setminus \{0\}, \\ \beta(h\bar{\theta}_0) & \text{if } \beta \in (\bar{\alpha}_0, \frac{5}{3}). \end{cases}$$

Again θ is a well-defined homomorphism. Then (ϕ, θ) extends to a homomorphism ψ of $G * H$ into $A(\Omega_0)$. Observe that $\psi|_{\Delta_n} = \phi_n|_{\Delta_n}$ for each $n \in \omega$ by the definitions of ϕ, θ , and ψ . Let $T = \{\alpha_0(f\psi) : f \in G * H\} \subseteq \Omega_0$. By the construction, $\alpha_n \in T$ for all $n \in \omega$, and as each $w \in (G * H) \setminus \{e\}$ is w_n for some $n \in \omega$ and $\alpha_n(w_n\psi) = \alpha_n(w_n\phi_n) \neq \alpha_n$, the natural map $\bar{\psi}$ obtained from ψ by restricting each $f\psi$ to T is an isomorphism of $G * H$ onto a transitive sublattice subgroup of $A(T)$. Since every element of $G\bar{\psi}$ fixes $-\frac{2}{9} \in T$ but $\sup \Gamma = -\frac{2}{9}$ and $\inf \Gamma = \bar{\alpha}_0$, the $(G * H)\bar{\psi}$ stabilizer of $-\frac{2}{9}$ has exactly one orbit in T below $-\frac{2}{9}$. Thus $(G * H)\bar{\psi}$ is a doubly transitive sublattice subgroup of $A(T)$ (see [2, Lemma 1.10.1]). Now T is a countable subset of $\mathbf{Q}$ and $A(T)$ is doubly transitive (since $(G * H)\bar{\psi}$ is). It follows that T must be order-isomorphic to the rational line, which proves the theorem. $\square$

Observe that if $e \neq w \in G * H$, then as $w, w^2, w^3, \dots$ are all distinct, $X = \{n \in \omega : w_n = w^m \text{ for some } 0 < m \in \omega\}$ is cofinal in ω . For each $n \in X$, $\alpha_n(w_n\bar{\psi}) = \alpha_n(w_n\psi) \neq \alpha_n$, whence $\alpha_n(w\bar{\psi}) \neq \alpha_n$ for each $n \in X$. Consequently the support of w is not bounded above in T ; that is, $(G * H)\bar{\psi}$ is pathological. (Aliter: In the enumeration of $G * H$, repeat each word infinitely often.)

4. The general case; Proof of Corollary 1'. Clearly it is enough to prove Corollary 1' for the free product of two nontrivial lattice-ordered groups, say G and H . Let $\{G_i : i \in I\}$ and $\{H_j : j \in J\}$ enumerate all the nontrivial countable (or just finitely generated) sublattice subgroups of G and H respectively and totally order $I \times J$. Let $\{w_{nij} : n = 1, 2, 3, \dots\}$ enumerate the nonidentity elements of the sublattice subgroup of $G * H$ generated by $G_i \cup H_j$ ($i \in I, j \in J$). There is a totally ordered field Ω containing isomorphic subintervals Ω_{ij} (with $\Omega_{i_1j_1} < \Omega_{i_2j_2}$ if $(i_1, j_1) < (i_2, j_2)$ in the total order on $I \times J$) such that $G * H$ can be embedded in the lattice-ordered group $A(\Omega_{ij}) = \text{Aut}(\langle \Omega_{ij}, \leq \rangle)$ for each $(i, j) \in I \times J$. As above, for each $(i, j) \in I \times J$, we can find bounded intervals Ω_{nij} of Ω_{ij} with $\Omega_{1ij} < \Omega_{2ij} < \dots$ and points $\alpha_{nij} \in \Omega_{nij}$ such that $\alpha_{nij}w_{nij} \neq \alpha_{nij}$ and if $w_{nij} = \bigvee_m \bigwedge_k w_{kmnij}$ with w_{kmnij} group words, then $\alpha_{nij}u_{kmnij} \in \Omega_{nij}$ for all initial subwords u_{kmnij} of w_{kmnij} . Indeed, just as in the previous section, we can map G and H into $A(\Omega_{ij})$ so that the extension ϕ_{ij} to $G * H$ gives $(G * H)\phi_{ij}$ doubly transitive on the orbit Δ_{ij} of α_{ij} and $w_{nij}\phi_{ij} \neq e$ for $n = 1, 2, 3, \dots$, for each $(i, j) \in I \times J$; moreover, $\alpha_{nij} \in \Delta_{ij}$ ($n = 1, 2, \dots, (i, j) \in I \times J$). Since each $w \in (G * H) \setminus \{e\}$ belongs to

the sublattice subgroup generated by $G_i \cup H_j$ for some $i \in I$ and $j \in J$, we obtain an embedding of $G * H$ into $A(\Delta)$, where $\Delta = \bigcup \{\Delta_{ij} : (i, j) \in I \times J\}$ with the inherited linear ordering.

Now if $G * H = A \times B$ with A and B nontrivial, let $e \neq a \in A$ and $e \neq b \in B$. Let $i \in I$ and $j \in J$ be such that a and b belong to the sublattice subgroup of $G * H$ generated by $G_i \cup H_j$. By the construction, $a\phi_{ij} \neq e \neq b\phi_{ij}$. But $(G * H)\phi_{ij} = A\phi_{ij} \times B\phi_{ij}$. Since $(G * H)\phi_{ij}$ is doubly transitive on Δ_{ij} , this is impossible (see [4]). Hence $G * H$ is directly indecomposable even in the category of groups.

If $e \neq w \in G * H$, then for some $(i, j) \in I \times J$, $w\phi_{ij} \neq e$. Since $(G * H)\phi_{ij}$ is doubly transitive on Δ_{ij} , there is $u \in G * H$ such that $[w\phi_{ij}, u\phi_{ij}] \neq e$ [2, Lemma 1.6.9]. Consequently, $[w, u] \neq e$; so w does not lie in the center of $G * H$. Therefore $G * H$ has trivial center. $\square$

5. Concluding remarks. Many of the results on free lattice-ordered groups to be found in [10] hold for free products of countable lattice-ordered groups by the same methods (only the doubly transitive representation is required). Rather than bore the reader, we refer him to [10, Corollaries 2, 3, 4, 6, 7, 8, and 9, Proposition 13 and, if G and H have strong order units (e.g., if they are finitely generated), Lemma 14(a), Theorem 15, and Corollary 16] for statements and proofs.

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