CHAOS AND HIGHER ORDER DIFFERENCES

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ABSTRACT. Asymptotic doubling of magnitude of higher order differences of a function is linked with abrupt character changes in the function.

1. Introduction and statement of results. Suppose that J is a connected open subset of the real numbers. If $\delta > 0$, $u \in J$, n is a positive integer, and f is a continuous real-valued function on J , then

$$\Delta_f(n; u, \delta) \equiv \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} f(u + k\delta)$$

provided that $v = u + n\delta \in J$. For each such u , n , δ one has the natural upper bound

$$|\Delta_f(n; u, \delta)| \leq 2^n |f|_{[u, v]},$$

i.e., $|\Delta_f(n; u, \delta)|^{1/n} \leq 2 |f|_{[u, v]}^{1/n}$ since $\sum_{k=0}^n \binom{n}{k} = 2^n$.

In this note we consider the implications of this last inequality being asymptotically close to equality. To be more precise, denote by G the collection of all bounded open intervals S such that $\bar{S} \subset J$. For f a continuous function on J , $S \in G$, and $\delta > 0$ so that $\delta < |S|$,

$$\Phi(S, \delta, f) \equiv \sup \left\{ |\Delta_f(n; u, \delta)|^{1/n} : u, u + n\delta \in S, n \in \mathbb{Z}^+ \right\}.$$

DEFINITION. The statement that f is chaotic at $x \in J$ means that if $S \in G$ and $x \in S$, then

$$\lim_{\delta \rightarrow 0} \Phi(S, \delta, f) = 2.$$

The following theorem helps place this definition into perspective.

THEOREM 1. If $S \in G$, then $\limsup_{\delta \rightarrow 0} \Phi(S, \delta, f) \leq 2$.

The definition is thus intended to describe certain extreme asymptotic behavior of higher order differences. It turns out that differences of a function which is chaotic at a point show a near doubling of size with increasing order of differences. Connections with work of Feigenbaum [2] seem to be of possible interest.

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The following shows that there is a wide class of examples of chaotic functions.

THEOREM 2. *Suppose $c \in J$, g is a real-analytic function on J , and f is a continuous function on J such that*

- (i) $f(x) = g(x)$ if $x \in J$ and $x \leq c$, and
- (ii) if $t > c$ there is x in (c, t) such that $f(x) \neq g(x)$.

Then f is chaotic at c .

The following result of Beurling gives some additional perspective on the definition.

THEOREM 3. *Suppose $S \in G$ and $\limsup_{\delta \rightarrow 0} \Phi(S, \delta, f) < 2$. Then f is real-analytic on S .*

The following is essentially a converse to the above.

THEOREM 4. *Suppose f is real-analytic at $c \in J$. There are $M, \epsilon > 0$ such that if $S \in G, c \in S$, and $|S| < \epsilon$, then*

$$\Phi(S, \delta, f) \leq |S|M \quad \text{for all } \delta < |S|.$$

The following indicates that differentiability is not necessary in order to have a nonchaotic function.

THEOREM 5. *Suppose $J = \mathbb{R}, 0 < a < 1, \lambda$ is an odd positive integer, $\lambda a \geq 1$, and*

$$f(x) = \sum_{k=0}^{\infty} a^k \cos(\lambda^k x), \quad x \in \mathbb{R}.$$

Then f is nowhere differentiable and also nowhere chaotic.

Such a function f is a Hardy-Weierstrass function (cf. [3]) which is known to be nowhere differentiable. On the other hand, if $f(x) = 0, x \leq 0, f(x) = \exp(-x^2), x > 0$, then f is infinitely differentiable yet chaotic at 0, according to the following theorem (on which Theorem 2 is based) [6, 5].

THEOREM 6. *Suppose f is a continuous function on $[0, 1]$ such that $f(x) = 0$ for $x \in [0, 1/2]$ and, if $t > 1/2$, there is x in $(1/2, t)$ such that $f(x) \neq 0$. Then if $a > 0$,*

$$\lim_{\delta \rightarrow 0} \left(\sup_{1-a \leq n\delta \leq 1} |\Delta_f(n; 0, \delta)| \right)^\delta = 2.$$

Some readers may find it interesting to do computer calculations concerning the above result (and consequently Theorem 2). To this end we offer the following Basic code:

```
10 INPUT D, A
20 Z = 0
30 FOR L = INT((1 - A)/D) TO INT(1/D)
40 Y = 0
50 M = 1
60 FOR J = 0 TO L
70 X = (L - J)*D
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80 IF X ← 1/2 THEN F = 0 ELSE F = X - 1/2
90 Y = Y + M * F
100 M = -M * (L - J) / (J + 1)
110 NEXT J
120 PRINT L, Y
130 IF Z < ABS(Y) THEN Z = ABS(Y)
140 NEXT L
150 PRINT Z ^ D
160 END

```

It is suggested that $D = .01$, $A = .1$ be tried. Line 80 contains a function definition and may be altered to compute with other functions.

2. Proofs.

PROOF OF THEOREM 1. Suppose $S \in G$, $u \in S$, $\delta > 0$, $n \in \mathbb{Z}^+$, and $u + n\delta \in S$. Then

$$\begin{aligned}
 |\Delta_f(n; u, \delta)| &= \left| \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} f(u + k\delta) \right| \\
 &\leq \sum_{k=0}^n \binom{n}{k} |f(u + k\delta)| \leq 2^n |f|_S
 \end{aligned}$$

and so

$$|\Delta_f(n; u, \delta)|^{1/n} \leq 2 |f|_S^{1/n}.$$

Hence

$$\Phi(S, \delta, f) = \sup \left\{ |\Delta_f(n; u, \delta)|^{1/n} : u, u + n\delta \in S, n \in \mathbb{Z}^+ \right\} \leq 2c^\delta,$$

where $c \equiv \max(1, |f|_S^{1/|S|})$. The conclusion follows since $\lim_{\delta \rightarrow 0} c^\delta = 1$.

PROOF OF THEOREM 4. Suppose f is real-analytic at $c \in J$. Pick $r, q \in \mathbb{R}$ such that $0 < q < r$ and r is less than the radius of convergence of a power series expansion of f about c . Denote by L, ρ positive numbers such that $|f^{(k)}(\alpha)| \leq Lk! \rho^k$ for all $\alpha \in [c - q, c + q]$, $k \in \mathbb{Z}^+$. Suppose also that $S \in G$, $c \in S$, $|S| < q$, $n \in \mathbb{Z}^+$, $\delta > 0$, and $u, u + n\delta \in S$. Then

$$|\Delta_f(n; u, \delta)| = |(\Delta_f(n; u, \delta)/\delta^n) \delta^n| = |f^{(n)}(\alpha)| \delta^n$$

for some $\alpha \in [u, u + n\delta]$. Hence

$$|\Delta_f(n; u, \delta)| \leq Ln! \rho^n \delta^n \leq Ln^n \rho^n \delta^n \leq |S|^n (\rho L^{1/n})^n \leq (|S|M)^n,$$

where $\beta \equiv \max(1, L)$, $M \equiv \rho\beta$. Therefore

$$\Phi(S, \delta, f) \leq |S|M \quad \text{if } 0 < \delta < |S|.$$

PROOF OF THEOREM 2. Suppose $c \in J$ and that f as given in Theorem 2 is not chaotic at c . Denote by S_1 a member of G containing c so that

$$\liminf_{\delta \rightarrow 0} \Phi(S_1, \delta, f) = 2 - \varepsilon$$

for some $\varepsilon > 0$. Denote by $\delta_1, \delta_2, \dots$ a decreasing sequence of positive numbers converging to 0 so that $\lim_{k \rightarrow \infty} \Phi(S_1, \delta_k, f) = 2 - \varepsilon$. Using Theorem 4, there is $S_2 \in G$ such that S_2 contains c and $\Phi(S_2, \delta, g) \leq \varepsilon/2$ if $0 < \delta < |S_2|$. Define h so that if $x \in J$ then

$$h(x) = \begin{cases} 0 & \text{for } x \leq c, \\ f(x) - g(x) & \text{for } x > c. \end{cases}$$

Note that if $t > x$ there is x_1 in (x, t) so that $h(x_1) \neq 0$. Note also that $f(t) = h(t) + g(t)$, $t \in J$. Pick r and s in $S_1 \cap S_2$ so that $r < c < s$ and $c = (r + s)/2$. Define w on $[0, 1]$ so that $w(t) = h(ts + (1 - t)r)$, $t \in [0, 1]$. Note that

$$\Delta_w(n; 0, \delta) = \Delta_h(n, r, \delta(s - r)), \quad n \in \mathbb{Z}^+, 0 < n\delta < 1.$$

Suppose $a > 0$ and note that by Theorem 5,

$$\lim_{k \rightarrow \infty} \left(\sup_{1-a \leq n\beta_k \leq 1} |\Delta_w(n; 0, \beta_k)| \right)^{\beta_k} = 2,$$

where $\beta_k \equiv \delta_k/(s - r)$, $k \in \mathbb{Z}^+$.

For each $k > 0$, pick n_k such that

$$|\Delta_w(n_k, 0, \beta_k)| = \sup_{1-a \leq n\beta_n \leq 1} |\Delta_w(n; 0, \beta_k)|$$

and note that

$$|\Delta_w(n_k; 0, \beta_k)|^{\beta_k} \rightarrow 2 \quad \text{as } k \rightarrow \infty.$$

Since $\beta_k n_k \leq 1$, $k \in \mathbb{Z}^+$, it follows that $1/n_k \geq \beta_k$, $k \in \mathbb{Z}^+$, and hence

$$|\Delta_w(n_k; 0, \beta_k)|^{1/n_k} \rightarrow 2 \quad \text{as } k \rightarrow \infty$$

since

$$\limsup_{k \rightarrow \infty} |\Delta_w(n_k; 0, \beta_k)|^{\beta_k} \leq \limsup_{k \rightarrow \infty} |\Delta_w(n_k; 0, \beta_k)|^{1/n_k} \leq 2.$$

Therefore, $|\Delta_h(n_k, r, \delta_k)|^{1/n_k} \rightarrow 2$ as $k \rightarrow \infty$. But

$$|\Delta_h(n_k; r, \delta_k)| \leq |\Delta_f(n_k; r, \delta_k)| + |\Delta_g(n_k; r, \delta_k)|$$

and so

$$|\Delta_h(n_k; r, \delta_k)|^{1/n_k} \leq (|\Delta_f(n_k; r, \delta_k)| + \varepsilon/2)^{1/n_k}.$$

Hence

$$\limsup_{k \rightarrow \infty} |\Delta_h(n_k; r, \delta_k)|^{1/n_k} \leq 2 - \varepsilon$$

since

$$\limsup_{k \rightarrow \infty} \Phi(S_1 \cap S_2, \delta_k, f) \leq 2 - \varepsilon, \quad \text{a contradiction.}$$

Hence f is chaotic at c and the theorem is established.

PROOF OF THEOREM 5. Under the hypothesis, write

$$f(x) = \operatorname{Re} \left(\sum_{k=0}^{\infty} a^k \exp(i\lambda^k x) \right), \quad x \in \mathbb{R}.$$

An easy calculation gives

$$\Delta_f(n; u, \delta) = \operatorname{Re} \left(\sum_{k=0}^{\infty} a^k \exp(i\lambda^k x) (\exp(i\lambda^k \delta) - 1)^n \right),$$

and hence

$$|\Delta_f(n; u, \delta)| \leq 2^n \sum_{k=0}^{\infty} a^k |\sin(\lambda^k \delta / 2)|^n, \quad u \in \mathbb{R}, \delta > 0, n \in \mathbb{Z}^+,$$

using the identity $\exp(i\alpha) - 1 = 2i \exp(i\alpha/2) \sin(\alpha/2)$, $\alpha \in \mathbb{R}$. Choose $\delta_q = 2\pi/\lambda^q$, $q \in \mathbb{Z}^+$. Clearly for $z, q \in \mathbb{Z}^+$, $\lambda^k \delta_q / 2 = \pi \lambda^{k-q} = (\pi/2)(2\lambda^{k-q})$. Hence if $k - q \geq 0$ then $\lambda^k \delta_q / 2$ is an even multiple of $\pi/2$ and so $\sin(\lambda^k \delta_q / 2) = 0$. On the other hand if $k - q < 0$ then $\lambda^k \delta_q / 2 = (\pi/2) 2/\lambda^{q-k}$ and $2/\lambda^{q-k} < 2/\lambda \leq 2/3$ and so $0 < \sin(\lambda^k \delta_q / 2) \leq \sin(\pi/3) = \sqrt{3}/2$. Hence

$$|\Delta_f(n; u, \delta_q)| \leq 2^n \sum_{k=0}^{\infty} a^k (\sqrt{3}/2)^n \leq (\sqrt{3})^n (1-a)^{-1}, \quad n, q \in \mathbb{Z}^+.$$

Denote by S a bounded open interval. Note that if $n, q \in \mathbb{Z}^+$ and $u, u + n\delta_q \in S$, then

$$|\Delta_f(n; u, \delta_q)|^{1/n} \leq \sqrt{3} (1-a)^{-1/n}.$$

Suppose now that $\limsup_{q \rightarrow \infty} \Phi(S, \delta_q, f) = 2$ and denote by $\{n_q\}_{q=1}^{\infty}$ a sequence in $\mathbb{Z}^+$ and by $\{u_q\}_{q=1}^{\infty}$ a sequence in S so that $u_q + \delta_q n_q \in S$, $q \in \mathbb{N}$, and

$$\limsup_{q \rightarrow \infty} |\Delta_f(n_q, u_q, \delta_q)|^{1/n_q} = 2.$$

It must be that $n_q \rightarrow \infty$ as $q \rightarrow \infty$ for otherwise $\{|\Delta_f(n_q, u_q, \delta_q)|^{1/n_q}\}_{q=1}^{\infty}$ would have a subsequence converging to 0 (using the continuity of f). Hence

$$|\Delta_f(n_q, u_q, \delta_q)|^{1/n_q} \leq \sqrt{3} (1-a)^{-1/n_q} \rightarrow \sqrt{3} \quad \text{as } q \rightarrow \infty,$$

a contradiction.

Therefore $\limsup_{q \rightarrow \infty} \Phi(S, \delta_q, f) < 2$ and hence f is not chaotic at any point of S . Hence f is nowhere chaotic.

Theorem 6 is essentially Lemma 8 of [5]. In connection with Theorem 3 see [1, 6].

3. Notes and comments. This work is closely related to early work of this writer on quasi-analytic collections [6, 7]. A collection of continuous functions on an open set J is called quasi-analytic if no two members of it agree on an open subset of J . It may be seen that a function which is nonchaotic at $c \in J$ is for some bounded open interval S containing c a member of the quasi-analytic collection $G(\sigma, S)$ (cf. [6]) where $\sigma = \{\delta_q\}_{q=1}^{\infty}$ so that $\limsup_{q \rightarrow \infty} \Phi(S, \delta_q, f) < 2$. It is mentioned that all members of all the quasi-analytic collections defined in [6 and 7] are nonchaotic.

For relationships between Theorems 3 and 5 with semigroups of operators, see [6, 4] as well as [1, 8].

Near doubling of differences in chaotic situations may be illustrated by running the Basic code given above for various continuous functions f which are zero on $[0, 1/2]$ but not identically zero on any interval $[0, t]$, $t > 1/2$. Without a fairly careful analysis of roundoff error one may falsely conclude that certain nonchaotic functions are chaotic (try the sine function for example). In a sense, roundoff error sometimes induces a chaos of its own which is easily confused with the mathematical chaos defined in this note. In fact observation of false period doubling in computations with Feigenbaum's function ($f_b: f_b(x) = bx(1 - x)$, $x \in [0, 1]$; cf. [2]) led to the idea of connecting [5–7] with the study of chaos.

This writer is indebted to Walter Rudin for pointing out reference [3] and for his general comments concerning lacunary Fourier series.

Finally this writer offers the outright speculation that chaos as defined here (but extended to higher dimensional cases) may provide a mathematical expression, in certain instances, of the physical phenomena of turbulence.

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