REGULARIZING PROPERTIES OF NONLINEAR SEMIGROUPS

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ABSTRACT. It is known that some classes of m -accretive operators A generate Lipschitz continuous semigroups of contractions; that is $\|S(t+h)x - S(t)x\| \leq L(\|x\|)h/t$, $0 \leq t \leq t+h \leq T$, $x \in \overline{D(A)}$. If the underlying Banach spaces X and X^* are uniformly convex and an m -accretive operator B is bounded, we prove, in particular, that the semigroup generated by $A + B$ is Hölder continuous. The proof is based on a result on the structure of accretive operators obtained via the Kuratowski-Ryll-Nardzewski Selection Theorem. Also, we consider some applications of these results to the existence of solutions of $u' + Au + Bu \ni Cu$, $u(0) = u_0$.

Let X be a Banach space with the norm $\|\cdot\|$. The duality mapping J from X into the class of subsets of its dual X^* is defined by $J(x) = \{z \in X^*: z(x) = \|x\|^2 = \|z\|^2\}$ for each $x \in X$. For each pair $x, y \in X$ we define

$$\langle y, x \rangle_s = \sup\{z(y): z \in J(x)\}.$$

A (multivalued) operator $A \subset X \times X$ is called accretive if $\langle y_1 - y_2, x_1 - x_2 \rangle_s \geq 0$ for any $(x_1, y_1), (x_2, y_2) \in A$. If the range $R(I + \lambda A) = X$ for some (equivalently for all) $\lambda > 0$, then the operator A is called m -accretive. (General works on this subject are, e.g., [1, 5, 7].)

Crandall's and Liggett's theorem [5] states that any m -accretive operator A generates a (nonlinear) semigroup of contractions $S(t): \overline{D(A)} \rightarrow \overline{D(A)}$, $t \geq 0$, by $S(t)x = \lim_{n \rightarrow \infty} [I + (t/n)A]^{-n}x$, $x \in \overline{D(A)}$. The function $u(t) = S(t)u_0$ can be considered as a generalized solution of the initial value problem

$$(1) \quad \begin{cases} u'(t) + Au(t) \ni 0, & t \geq 0, \\ u(0) = u_0 \in \overline{D(A)} \end{cases}$$

(see, e.g., [1, §3.1; 5]).

Generally, little can be said on the regularity of the function $u(t)$. In particular, u can be nondifferentiable, or, even, not Lipschitz continuous. However, for some classes of m -accretive operators the generated semigroup $S(t)$ behaves better. For example, if $S(t)$ is generated by the subdifferential of a convex, lower-semicontinuous function $\phi(x)$ on a Hilbert space X , then $\|S(t+h)x - S(t)x\| \leq L(\|x\|)h/t$, $0 \leq t \leq t+h \leq T$, $x \in \overline{D(A)}$; where $T > 0$ and $L(r)$ is a nondecreasing function [3; 1, §4.2].

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The semigroups generated by homogeneous operators have similar behavior (see [4, Theorem 1]).

The main purpose of this note is to study such regularizing properties for the solutions of the perturbed initial value problem

$$(2) \quad \begin{cases} u'(t) + Au(t) + Bu(t) \ni 0, \\ u(0) = u_0, \quad 0 \leq t \leq T, \end{cases}$$

where A is an m -accretive operator and $B: X \rightarrow 2^X$ is a bounded (multivalued) operator. (B is bounded if $\sup\{\|y\|: y \in Bx, \|x\| \leq r\}$ is finite for any $r > 0$.) To define a solution of (2) we recall [2; 1, 3.2] that for any integrable function $f \in L^1(0, T; X)$, $T > 0$, there exists a unique integral solution of the initial value problem

$$(3) \quad \begin{cases} u'(t) + Au(t) \ni f(t), \quad 0 \leq t \leq T, \\ u(0) = u_0. \end{cases}$$

That is a continuous function $u: [0, T] \rightarrow X$ such that

$$(4) \quad \frac{1}{2}\|u(t) - x\|^2 \leq \frac{1}{2}\|u(s) - x\|^2 + \int_s^t \langle f(\tau) - y, u(\tau) - x \rangle_s d\tau$$

for any $0 \leq s \leq t \leq T$ and $[x, y] \in A$. A continuous function u is called the strong solution of (3) if u is continuous, differentiable a.e. on $[0, T]$, and (3) is satisfied a.e. on $[0, T]$ (we use only the Lebesgue measure on $[0, T]$).

We define the operator $H: L^1(0, T; X) \rightarrow C([0, T]; X)$ by $Hf = u$, where u is the unique integral solution of (3). It is known [7, Remark 8.4] that

$$(5) \quad \|Hf(t) - Hg(t)\| \leq \|Hf(s) - Hg(s)\| + \int_s^t \|f(\tau) - g(\tau)\| d\tau$$

for $0 \leq s \leq t \leq T$.

DEFINITION 1. A continuous function $u: [0, T] \rightarrow \overline{D(A)}$ is called a solution of (2) if there exists a measurable function $f \in L^1(0, T; X)$ such that $f(t) \in -Bu(t)$ for a.e. $t \in [0, T]$ and $Hf = u$.

THEOREM. Let X be a Banach space, $T > 0$, A be an m -accretive operator which generates a continuous semigroup of contractions $S(t)$, $t \geq 0$. Suppose that a (multivalued) operator $B: X \rightarrow 2^X$ is bounded and $D(B) \subset \overline{D(A)}$. If there exists $\gamma > 0$, $0 < \mu \leq 1$, and a nondecreasing function $L: [0, \infty) \rightarrow [0, \infty)$ such that $\|S(t+h)x - S(t)x\| \leq t^{-\gamma}L(\|x\|)h^\mu$ for any $x \in \overline{D(A)}$, $0 \leq t \leq t+h \leq T$, then any solution of (2) satisfies

$$\|u(t+h) - u(t)\| \leq C(\mu, \gamma, T, M, N)t^{-\gamma}h^{\mu/(\gamma+1)}, \quad 0 \leq t \leq t+h \leq T,$$

where $N = \max\{\|u(t)\|, 0 \leq t \leq T\}$ and $M = \sup\{\|y\|: y \in Bx, \|x\| \leq N\}$.

This Theorem follows immediately from the above definition and the following lemma.

LEMMA 1. Suppose that $f \in L^1(0, T; X)$ and $\sup_{0 \leq t \leq T}\|f(t)\| \leq M$. If the operator A satisfies the conditions of the Theorem then the integral solution $u(t)$ of (3) satisfies

$$\|u(t+h) - u(t)\| \leq C(\mu, \gamma, T, M, N)t^{-\gamma}h^{\mu/(\gamma+1)},$$

where the constants are as in the Theorem.

PROOF. Fix $t > 0$. For any $0 \leq h \leq T - t$ and $0 \leq \sigma < t$ we have (by (5))

$$\begin{aligned} \|u(t+h) - u(t)\| &\leq \|u(t+h) - S(t-\sigma+h)u(\sigma)\| \\ &\quad + \|S(t-\sigma+h)u(\sigma) - S(t-\sigma)u(\sigma)\| + \|S(t-\sigma)u(\sigma) - u(t)\| \\ &\leq M(t-\sigma+h) + (t-\sigma)L(\|u(\sigma)\|)h + M(t-\sigma). \end{aligned}$$

Thus $\|u(t+h) - u(t)\| \leq \min_{0 \leq \sigma < t} [2Mt + Mh - 2M\sigma + L(N)(t-\sigma)^{-\gamma}h^\mu]$.
Therefore

$$\begin{aligned} \|u(t+h) - u(t)\| &\leq h^{\mu/(\gamma+1)} \left[Mh^{1-\mu/(\gamma+1)} + (2M)^{\gamma/(\gamma+1)} \right. \\ &\quad \left. + (L(N))^{1/(\gamma+1)}(\gamma+1)\gamma^{-\gamma/(\gamma+1)} \right] \\ &\leq h^{\mu/(\gamma+1)} C_1(\mu, \gamma, T, M, N) \end{aligned}$$

for $0 < h < ((2M/\gamma L(N))t^{\gamma+1})^{1/\mu}$ and

$$\begin{aligned} \|u(t+h) - u(t)\| &\leq t^{-\gamma} [2Mt^{\gamma+1} + Mht^\gamma + L(N)h^\mu] \\ &\leq t^{-\gamma} [L(N)(\gamma+1)h^\mu + Mt^\gamma h] \leq t^{-\gamma} h^{\mu/(\gamma+1)} C_2(\mu, \gamma, T, M, N) \end{aligned}$$

for $h \geq ((2M/\gamma L(N))t^{\gamma+1})^{1/\mu}$. Thus

$$\begin{aligned} \|u(t+h) - u(t)\| &\leq h^{\mu/(\gamma+1)} C_1 + t^{-\gamma} h^{\mu/(\gamma+1)} C_2 \\ &\leq t^{-\gamma} h^{\mu/(\gamma+1)} C(\mu, \gamma, T, M, N) \end{aligned}$$

and the proof of the lemma is complete.

It was assumed in the Theorem that the initial value problem (2) has a solution on $[0, T]$. It is known (see, e.g., [1, 3.3]) that such a solution exists for different assumptions on the operators A , B and the underlying Banach space X . If the operator A is homogeneous and m -accretive and B is Lipschitz continuous, a stronger regularity result can be found in [4]. We are interested in the case where both operators A and B are m -accretive and not necessarily continuous. In the sequel we suppose that both X and X^* are uniformly convex and separable. Following [1, Chapters II and III] we define $|Ax| = \inf\{\|y\|: y \in Ax\}$ and $A^0x = \{y \in Ax: \|y\| = |Ax|\}$. It is known [1, p. 118] that in this case the operator A^0 is single-valued and $(d/dt)S(t)u_0 = A^0S(t)u_0$, a.e. $t > 0$, $u_0 \in D(A)$.

If we suppose that $D(A) \subset D(B)$, and B is bounded, then [1, Theorem 2.3.5] the operator $A + B$ is m -accretive. We denote the correspondent semigroup by $S_{A+B}(t)$. The following lemma handles the structure of the operator $A + B$ via the Kuratowski-Ryll-Nardzewski Selection Theorem (see the appendix).

LEMMA 2. *Let X, X^* be uniformly convex, separable Banach spaces and let $T > 0$. Let operators B, A be m -accretive, B be bounded, and $D(B) \supset D(A)$. Then $u(t) = S_{A+B}(t)u_0$, $t \geq 0$, $u_0 \in D(A)$, is a solution of the system $u' + Au + Bu \ni 0$, $u(0) = u_0$, in the sense of Definition 1.*

PROOF. We should prove that there exists an integrable function $f \in L^1(0, T; X)$, $f(t) \in -Bu(t)$ a.e. $t \in [0, T]$ such that $Hf = u$. Note that $|Au_0| < \infty$ since $u_0 \in D(A)$. Therefore $|(A+B)u_0| < \infty$ and $u(t)$ is the strong solution, that is $u'(t)$ exists for almost every $t \in [0, T]$ and $u'(t) + Au + Bu(t) \ni 0$ a.e. on $[0, T]$.

By [1, Theorem 3.1.6] $u'(t) = (A + B)^0 u(t)$ a.e. $t \in [0, T]$. Since $(A + B)^0 z = \lim_{\lambda \rightarrow 0} (A + B)_\lambda z$ for each $z \in D(A + B)$ ($(A + B)_\lambda$ is the Yosida approximation; see [1, Proposition 2.3.6]) we get that $u'(t)$ is integrable as the pointwise limit of the uniformly bounded continuous functions $(A + B)_\lambda u(t)$. By Lemma 3 in the Appendix there exist measurable functions g and f such that $-g(t) + f(t) = u'(t)$, $g(t) \in Au(t)$, $-f(t) \in Bu(t)$ a.e. $t \in [0, T]$. These functions are integrable, since they are bounded. Thus $u'(t) + g(t) = f(t)$ or $u'(t) + Au(t) \ni f(t)$ a.e. $t \in [0, T]$, $u(0) = u_0$. Since $u(t)$ is the strong solution of the above system it is also its integral solution.

COROLLARY 1. *Let X, X^* be uniformly convex, separable Banach spaces and $T > 0$. Let the operators B and A be m -accretive and let B be bounded. If $D(B) \supset \overline{D(A)}$ and the semigroup $S_A(t)$ satisfies $\|S_A(t+h)x - S_A(t)x\| \leq t^{-\gamma} L(\|x\|)h^\mu$ for any $x \in \overline{D(A)}$, $0 \leq t \leq t+h \leq T$ ($\gamma > 0$, $0 < \mu \leq 1$, L is nondecreasing), then the semigroup $S_{A+B}(t)$ satisfies*

$$\|S_{A+B}(t+h)u_0 - S_{A+B}(t)u_0\| \leq Ct^{-\gamma}h^{\mu/(\gamma+1)},$$

where $0 \leq t \leq t+h \leq T$, $u_0 \in \overline{D(A)}$, and the constant C is chosen as in the Theorem.

PROOF. It is enough to prove the assertion for any $u_0 \in D(A)$. By Lemma 2 $u(t) = S_{A+B}(t)u_0$ is the solution of (2) in the sense of Definition 1. By the Theorem it satisfies the required inequality.

REMARK. A detailed analysis of the proof shows that it is enough to require $D(A) \subset D(B)$ in the above corollary.

COROLLARY 2. *Let X, X^* be uniformly convex, separable Banach spaces, A be a homogeneous m -accretive operator (see [4]), and B be a bounded, everywhere defined in X m -accretive operator. Suppose that $C: X \rightarrow X$ is a compact continuous operator. Then the initial value problem $u'(t) + Au(t) + Bu(t) \ni Cu(t)$, $0 \leq t \leq T$, $u(0) = u_0 \in \overline{D(A)} \subset X$, has a local solution.*

PROOF. Since A is homogeneous the generated semigroup $S_A(t)$, $t \geq 0$, satisfies $\|S(t+h)x - S(t)x\| \leq L(\|x\|)t^{-1}h$ for any $x \in \overline{D(A)}$, $0 \leq t \leq t+h \leq T$ (see [4, Theorem 1]). By Corollary 1 the semigroup $S_{A+B}(t)$ satisfies

$$\|S_{A+B}(t+h)u_0 - S_{A+B}(t)u_0\| \leq Ct^{-1}h^{1/2}, \quad u_0 \in \overline{D(A)}.$$

Therefore this semigroup S_{A+B} is equicontinuous (that is the family of functions $\{t \rightarrow S_{A+B}(t)x, x \in \overline{D(A+B)} \cap G\}$ is equicontinuous on $(0, T)$ for any bounded subset $G \subset X$). By [8, Proposition 1.2] the initial value problem has a local solution.

Appendix. We will use the Kuratowski-Ryll-Nardzewski Selection Theorem, which we will restate as follows:

THEOREM [6, p. 286]. *Let Y be a complete, separable metric space. Suppose $F: [0, T] \rightarrow 2^Y$ is such that $F(t) \subset Y$ is closed for any $t \in [0, T]$, and for each open set V in Y the set $\{\omega \in [0, T]: F(\omega) \cap V \neq \emptyset\}$ is measurable. Then there exists a measurable function $f: [0, T] \rightarrow Y$ such that $f(\omega) \in F(\omega)$ for each $\omega \in [0, T]$.*

Recall that if a separable Banach space X and its dual X^* are uniformly convex, then any convex, bounded, and closed subset $W \subset X$ is weakly compact [9, §5.2]. The set W with the weak topology is a complete compact metric space.

LEMMA 3. *Let X, X^* be uniformly convex and separable Banach spaces. Suppose that $T > 0$; A, B are m -accretive operators in X , $D(A) \subset D(B)$, and B is bounded. Let $u: [0, T] \rightarrow D(A)$ be continuous and $h: [0, T] \rightarrow X$ be measurable and bounded.*

If $h(t) \in (A + B)u(t)$ almost everywhere on $[0, T]$ then there exist measurable functions g and $f: [0, T] \rightarrow X$ such that $g(t) \in Au(t)$, $f(t) \in Bu(t)$, and $g(t) + f(t) = h(t)$ a.e. on $[0, T]$.

PROOF. Define $F: [0, T] \rightarrow X \times X$ by $F(t) = \{(x, y) \in X \times X: x + y = h(t), x \in Au(t), y \in Bu(t)\}$. Note that $F(t) \neq \emptyset$ for any $t \in [0, T]$. Let $M_1 = \sup\{\|z\|: z \in Bu(t), t \in [0, T]\}$, $M_2 = \sup\{\|h(t)\|: t \in [0, T]\}$, and $M = M_1 + M_2$. Then $\|x\|, \|y\| \leq M$ for any $(x, y) \in F(t)$, $0 \leq t \leq T$. Hence $F(t) \subset Y = Z \times Z$, where $Z = \{z \in X: \|z\| \leq M\}$. We supply Z with the weak topology of X . Thus Z is a compact (complete) metrizable space and so is Y with an appropriate product topology. Let $Q \subset [0, T]$ be any closed subset such that $h|_Q$ is continuous. Let W be any closed subset of Y . Then the set $\{\omega \in Q: F(\omega) \cap W \neq \emptyset\}$ is closed in $[0, T]$. Indeed, if $t_n \in Q$, $t_n \rightarrow t$ as $n \rightarrow \infty$ and $(x_n, y_n) \in F(t_n)$, then we can assume (passing to a subsequence) that $x_n \rightarrow x$, $y_n \rightarrow y$ in Z , since Z is compact. That is $x_n \rightarrow x$, $y_n \rightarrow y$ weakly in X . Since the operators A and B are demiclosed [1, Proposition 2.3.5] we have $x \in Au(t)$ and $y \in Bu(t)$. Also $x + y = h(t)$ since for any functional $e \in X^*$ we get $e(x + y) = \lim_{n \rightarrow \infty} e(x_n + y_n) = \lim_{n \rightarrow \infty} e(h(t_n)) = e(h(t))$. Therefore the set $\{\omega \in [0, T]: F(\omega) \cap W \neq \emptyset\}$ is measurable in $[0, T]$ as the union of the above closed sets. Since any open set V in Y can be represented as a countable union of closed sets W_n we get that $\{\omega \in [0, T]: F(\omega) \cap V \neq \emptyset\}$ is measurable in $[0, T]$ and the Kuratowski-Ryll-Nardzewski Selection Theorem can be applied to the multivalued function $F: [0, T] \rightarrow Y$ to obtain measurable functions $g(t)$ and $f(t)$ such that $g(t) + f(t) = h(t)$, $g(t) \in Au(f)$, and $f(t) \in Bu(t)$. The functions g and f are measurable as functions into Z . Since the Banach space X is separable they are measurable as functions into X with its strong topology by the Pettis theorem [9, §5.4]

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