**$A \geq B \geq 0$ ASSURES $(B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$
FOR $r \geq 0$, $p \geq 0$, $q \geq 1$ WITH $(1+2r)q \geq p+2r$**

TAKAYUKI FURUTA

*Dedicated to Professor Masahiro Nakamura
with respect and affection*

ABSTRACT. An operator means a bounded linear operator on a Hilbert space. This paper proves the assertion made in its title. Theorem 1 yields the famous result that $A \geq B \geq 0$ assures $A^\alpha \geq B^\alpha$ for each $\alpha \in [0, 1]$ when we put $r = 0$ in Theorem 1. Also Corollary 1 implies that $A \geq B \geq 0$ assures $(BA^p B)^{1/p} \geq B^{(p+2)/p}$ for each $p \geq 1$ and this inequality for $p = 2$ is just an affirmative answer to a conjecture posed by Chan and Kwong. We cite three counterexamples related to Theorem 1 and Corollary 1.

1. Statement of results. In this paper, a capital letter means a bounded linear operator on a Hilbert space. It is well known that $A \geq B \geq 0$ does not always assure $A^2 \geq B^2$ in general. The purpose of this paper is to show "order preserving inequalities" on A and B in the case $A \geq B \geq 0$.

THEOREM 1. *If $A \geq B \geq 0$, then for each $r \geq 0$*

$$(i) \quad (B^r A^p B^r)^{1/q} \geq B^{(p+2r)/q}$$

and

$$(ii) \quad A^{(p+2r)/q} \geq (A^r B^p A^r)^{1/q}$$

hold for each p and q such that $p \geq 0$, $q \geq 1$ and $(1+2r)q \geq p+2r$.

COROLLARY 1. *If $A \geq B \geq 0$, then for each $r \geq 0$*

$$(i) \quad (B^r A^p B^r)^{1/p} \geq B^{(p+2r)/p}$$

and

$$(ii) \quad A^{(p+2r)/p} \geq (A^r B^p A^r)^{1/p}$$

hold for each $p \geq 1$.

2. Proofs of the results. First of all, we cite the following results.

THEOREM A [2]. *Let X and Y be bounded linear operators on a Hilbert space. We suppose that $X \geq 0$ and $\|Y\| \leq 1$. If f is an operator monotone function defined on $[0, \infty[$, then $f(Y^*XY) \geq Y^*f(X)Y$.*

THEOREM B [3]. *If S and T are bounded positive linear operators such that $S \geq T \geq 0$, then $S^\alpha \geq T^\alpha$ for each $\alpha \in [0, 1]$.*

PROOF OF THEOREM 1. In the case $1 \geq p \geq 0$, if $A \geq B \geq 0$, then $A^p \geq B^p$ by Theorem B, wh $(B^r A^p B^r)^{1/q} \geq (B^r B^p B^r)^{1/q} = B^{(p+2r)/q}$ holds for each

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$r \geq 0$ and $q \geq 1$ by Theorem B and $(1+2r)q \geq 1+2r \geq p+2r$ since $q \geq 1$ and $1 \geq p \geq 0$, so we have only to consider the case $p \geq 1$. We have only to show the following for each $r \geq 0$ and $p \geq 1$:

$$(1) \quad (B^r A^p B^r)^{(1+2r)/(p+2r)} \geq B^{1+2r}$$

since (i) of Theorem 1 for values q larger than $(p+2r)/(1+2r)$ follows by Theorem B. If $A \geq B \geq 0$, then $A + \varepsilon \geq B + \varepsilon$ for any $\varepsilon > 0$, so $B + \varepsilon$ and $A + \varepsilon$ are both invertible, therefore we may assume that A and B are invertible. In the case $0 \leq r \leq 1/2$, $A \geq B \geq 0$ assures $A^{2r} \geq B^{2r}$ by Theorem B, so $B^r A^{-2r} B^r \leq 1$, namely $\|A^{-r} B^r\| \leq 1$. Put $q = (p+2r)/(1+2r) \geq 1$. By Theorem A, we have

$$\begin{aligned} (B^r A^p B^r)^{1/q} &= (B^r A^{-r} A^{p+2r} A^{-r} B^r)^{1/q} \\ &\geq B^r A^{-r} A^{(p+2r)/q} A^{-r} B^r = B^r A B^r \geq B^{1+2r}. \end{aligned}$$

Put $A_1 = (B^r A^p B^r)^{1/q}$ and $B_1 = B^{1+2r}$. Then this inequality $A_1 \geq B_1$ means that (1) holds for $0 \leq r \leq 1/2$. Repeating (1) again for $0 \leq r_1 \leq 1/2$ and $p_1 \geq 1$

$$(B_1^{r_1} A_1^{p_1} B_1^{r_1})^{1/q_1} \geq B_1^{1+2r_1}$$

for $q_1 = (p_1 + 2r_1)/(1 + 2r_1)$; that is,

$$\{B^{(1+2r)r_1} (B^r A^p B^r)^{p_1/q} B^{(1+2r)r_1}\}^{1/q_1} \geq B^{(1+2r)(1+2r_1)}.$$

Put $p_1 = q$. Then we have

$$(2) \quad \{B^{(1+2r)r_1+r} A^p B^{r+(1+2r)r_1}\}^{1/q_1} \geq B^{(1+2r)(1+2r_1)}.$$

Put $r_2 = (2r+1)r_1 + r$. Then $q_1 = (p_1 + 2r_1)/(1 + 2r_1) = (p + 2r_2)/(1 + 2r_2)$ since $p_1 = q$ and $(1+2r)(1+2r_1) = 2r_2 + 1$. Consequently (2) means that (1) holds for $r_2 \in [0, 3/2]$ since $r, r_1 \in [0, 1/2]$ and, repeating this method, (1) holds for each $r \geq 0$ and (i) is shown. (ii) is easily obtained by (i) as follows. Without loss of generality, we may assume that B^{-1} exists. By hypothesis, $B^{-1} \geq A^{-1} \geq 0$. Then by (i), for each $r \geq 0$, $(A^{-r} B^{-p} A^{-r})^{1/q} \geq A^{-(p+2r)/q}$ holds for each p and q such that $p \geq 0$, $q \geq 1$, and $(1+2r)q \geq p+2r$. Taking inverses gives (ii), so the proof is complete.

PROOF OF COROLLARY 1. Put $q = p \geq 1$ in Theorem 1.

REMARK 1. The following conjecture is cited in [1].

CONJECTURE. If $A \geq B \geq 0$, then $(BA^2B)^{1/2} \geq B^2$ and $A^2 \geq (AB^2A)^{1/2}$.

Corollary 1 for $r = 1$ and $p = 2$ is just an affirmative answer to this conjecture. Moreover we can show a stronger result as follows. If $A \geq B \geq 0$, then $(BA^2B)^{3/4} \geq B^3$ and $A^3 \geq (AB^2A)^{3/4}$ putting $r = 1$, $p = 2$, and $q = 4/3$ in Theorem 1 because taking $2/3$ exponents of both sides of these inequalities, we have the exponents in the conjecture by Theorem B.

3. Counterexamples.

COUNTEREXAMPLE 1. In Theorem 1, $(1+2r)q \geq p+2r$ is essential. We cite a counterexample to Theorem 1 for p, q , and r without this condition. Put $r = 1$, $p = 3$, and $q = 1.5$. Then $(1+2r)q \not\geq p+2r$. Take B and C as

$$B = \begin{pmatrix} 4 & 5 \\ 5 & 10 \end{pmatrix}^2 \geq 0, \quad C = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}^2 \geq 0.$$

Then $A = B + C \geq B \geq 0$, but the computer shows

$$D = (BA^3B)^{2/3} - B^{10/3} = \begin{pmatrix} 4496.22169\dots & 7806.21196\dots \\ 7806.21196\dots & 13552.32949\dots \end{pmatrix}.$$

The eigenvalues of D are $-0.14778\dots$ and $18048.69897\dots$. Hence $(BA^3B)^{2/3} \not\geq B^{10/3}$.

COUNTEREXAMPLE 2. In Corollary 1, $p \geq 1$ is essential. We cite a counterexample for $1 > p \geq 0$. Put $p = 1/2$ and $r = 1$ in Corollary 1. Take A and B as

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}.$$

Obviously $A \geq B \geq 0$ and the computer shows

$$C = (BA^{1/2}B)^2 - B^5 = \begin{pmatrix} 7.12132\dots & -13.62132\dots \\ -13.62132\dots & 25.56066\dots \end{pmatrix}.$$

The eigenvalues of C are $-0.10719\dots$ and $32.78917\dots$. Consequently $(BA^{1/2}B)^2 \not\geq B^5$. Also this turns out to be a counterexample for $p = 0.9$ and we omit to detail it.

COUNTEREXAMPLE 3. Closely related to Corollary 1, we consider that the question is: If $A \geq B \geq 0$, then does

$$(*) \quad (B^r A^p B^r)^{1/p} \geq B^{r/p} A B^{r/p}$$

hold for each $p \geq 1$. If $(*)$ holds for each $p \geq 1$, Corollary 1 is easily shown because

$$(B^r A^p B^r)^{1/p} \geq B^{r/p} A B^{r/p} \geq B^{(p+2r)/p}$$

whenever $A \geq B \geq 0$. But we cite a counterexample to $(*)$ for $p = 2$ and $r = 1$ as follows. Take A and B as

$$A = \begin{pmatrix} 10 & -5 \\ -5 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 10 & -5 \\ -5 & 5 \end{pmatrix}.$$

Obviously $A \geq B \geq 0$ and the computer shows

$$C = (BA^2B)^{1/2} - B^{1/2}AB^{1/2} = \begin{pmatrix} 5.04042\dots & 1.24937\dots \\ 1.24937\dots & -4.01746\dots \end{pmatrix}.$$

The eigenvalues of C are $-4.18663\dots$ and $5.20959\dots$. Consequently $(BA^2B)^{1/2} \not\geq B^{1/2}AB^{1/2}$.

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DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, HIROSAKI UNIVERSITY,
BUNKYO-CHO 3, HIROSAKI 036, AOMORI, JAPAN