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# KNOTS WITH FINITE WEIGHT COMMUTATOR SUBGROUPS

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ABSTRACT. An example of a knot in $S^3$ is constructed which has a companion of winding number zero but for which the commutator subgroup of the fundamental group of the complement is of finite weight. This provides a counterexample to a conjecture made by Jonathan Simon.

A conjecture made by J. Simon states: If a knot $K \subset S^3$ has a companion of winding number zero, then the commutator subgroup of $\pi_1(S^3 - K)$ is of infinite weight. The conjecture appears in Problem 1.14 of Kirby's problem list **[K]**. This paper presents a counterexample. The knot $J \subset S^1 \times B^2$ illustrated in Figure 1 has the property that for any embedding $f\colon S^1 \times B^2 \to S^3$, the commutator subgroup of $\pi_1(S^3 - f(J))$ is of finite weight. (In Figure 1, $S^1 \times B^2$ is viewed as the complement in $\mathbf{R}^3$ of the $z$-axis, indicated by the $\otimes$, in a projection onto the $x$-$y$ plane.)

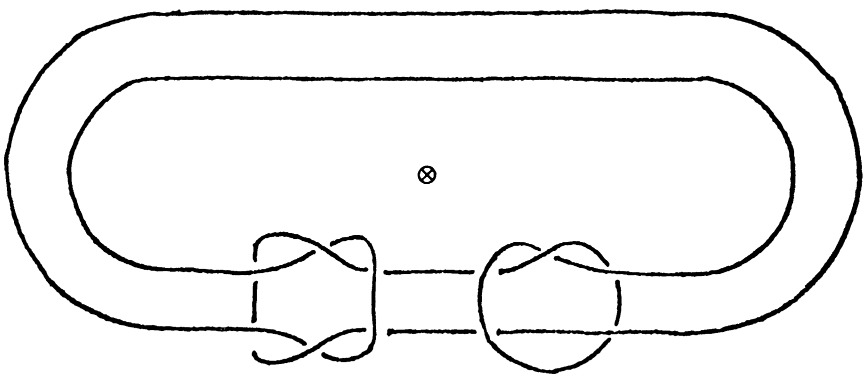

FIGURE 1

**Preliminaries.** A reference for the results and techniques of knot theory used here is Rolfsen's book **[R]**.

A knot $K \subset S^3$ has a companion if there is an embedding $f$ of $S^1 \times B^2$ into $S^3$ with $K \subset f(S^1 \times B^2)$ and with $f(S^1 \times \partial B^2)$ incompressible in $S^3 - K$. In this case $f(S^1 \times \{0\})$ is called a companion of $K$. The winding number of the companion is the homology class in $H_1(f(S^1 \times B^2)) \cong Z$ represented by $K$.

The weight of a group is the minimum number of elements which can normally generate it.

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To prove that the commutator subgroup of a knot group has finite weight, we will consider the following situation. $F$ will denote an oriented Seifert surface for a knot $K \subset S^3$. The positive and negative pushoffs of $F$ into $S^3 - F$ will be denoted by $i_+$ and $i_-$.

PROPOSITION. *If both* $\pi_1(S^3 - F)/\langle i_+(\pi_1(F))\rangle$ *and* $\pi_1(S^3 - F)/\langle i_-(\pi_1(F))\rangle$ *are trivial, then the commutator subgroup of* $\pi_1(S^3 - K)$ *is of finite weight.*

PROOF. The commutator subgroup of $\pi_1(S^3 - K)$ is isomorphic to $\pi_1(\widetilde{S^3 - K})$, where $\widetilde{S^3 - K}$ denotes the infinite cyclic cover of $S^3 - K$. Let $N(F)$ denote an open regular neighborhood of $F$. Then $\widetilde{S^3 - K}$ is homeomorphic to the infinite union

$$\cdots \cup (S^3 - N(F)) \cup (S^3 - N(F)) \cup (S^3 - N(F)) \cup \cdots$$

where $i_-(F)$ in each copy of $S^3 - N(F)$ is identified with $i_+(F)$ in the next copy.

This decomposition of $\widetilde{S^3 - K}$ induces a decomposition of $\pi_1(\widetilde{S^3 - K})$ as

$$\cdots * G_{-2} \underset{H_{-2}}{*} G_{-1} \underset{H_{-1}}{*} G_0 \underset{H_0}{*} G_1 \underset{H_1}{*} G_2 \underset{H_2}{*} \cdots$$

with each $G_i \cong \pi_1(S^3 - F)$ and $H_i \cong \pi_1(F)$. We are not assuming that $i_+$ or $i_-$ induce injections on $H_i$.

Any element in $G_n$ is a product of conjugates of elements in $i_+(H_{n-1})$, since $G_n/\langle i_+(H_{n-1})\rangle = 1$. Hence any element in $G_n$ is the product of conjugates of elements in $i_-(H_{n-1}) \subset G_{n-1}$. Similarly, any element in $G_n$ is the product of conjugates of elements in $G_{n+1}$. Proceeding by induction, any element in $\pi_1(\widetilde{S^3 - K})$ is the product of conjugates of elements in $G_0$. Since $\pi_1(S^3 - F) = G_0$ is finitely generated, $\pi_1(\widetilde{S^3 - K})$ is of finite weight.

$\pi_1(S^3 - f(J))$ **has finite weight commutator subgroup.** Figure 2 provides a second illustration of $J \subset S^1 \times B^2$. A Seifert surface $F$ consists of four bands joined as indicated in the figure. $F$ is oriented so that at $\alpha \cap \beta$ the positive normal points at

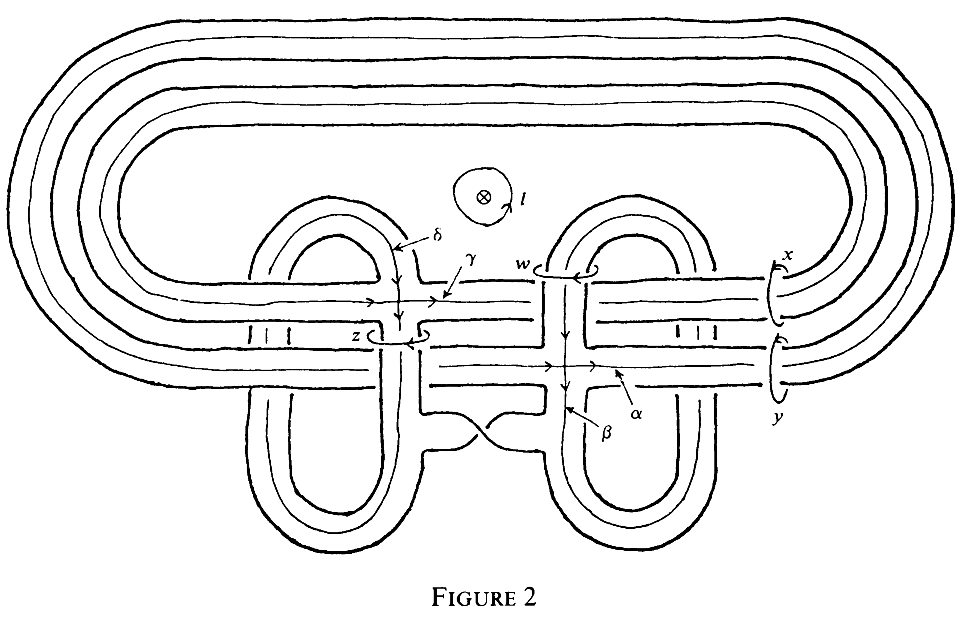



FIGURE 2

the viewer. A set of normal generators for $\pi_1(S^1 \times B^2 - F)$, $\{1, x, y, z, w\}$, is indicated, as is a set of loops on $F$, $\{\alpha, \beta, \gamma, \delta\}$.

Let $f$ be an embedding of $S^1 \times B^2$ into $S^3$ with $f(S^1 \times \{0\})$ representing a nontrivial knot $K$. The torus $f(S^1 \times \partial B^2)$, which we will later see is incompressible in $S^3 - f(J)$, induces a decomposition

$$\pi_1(S^3 - f(F)) \cong \pi_1(S^1 \times B^2 - F) \underset{Z^2}{*} \pi_1(S^3 - K).$$

Denote $i_+(\pi_1(f(F)))$ by $N_+$. To apply the proposition we need to show that $\pi_1(S^3 - f(F))/\langle N_+ \rangle$ is trivial.

We observe first that the meridian to $K$ is trivial in this quotient group, as it is represented by $i_+(\beta)$. Since the meridian to $K$ normally generates $\pi_1(S^3 - K)$, we have $\pi_1(S^3 - K) \subset \langle N_+ \rangle$, and in particular, $l \in \langle N_+ \rangle$.

It now suffices to show that $\pi_1(S^1 \times B^2 - F)/\langle i_+(\pi_1(F)), l \rangle$ is trivial. By considering $i_+(\alpha)$, $i_+(\beta)$, $i_+(\gamma)$, and $i_+(\delta)$ we see that in addition to $l$, the four elements $lz$, $yx$, $wlz$, and $y$ are trivial in this quotient group. It follows immediately that $l, x, y, z$, and $w$ are all trivial in the quotient. As these normally generate $\pi_1(S^1 \times B^2 - F)$, the quotient group is trivial.

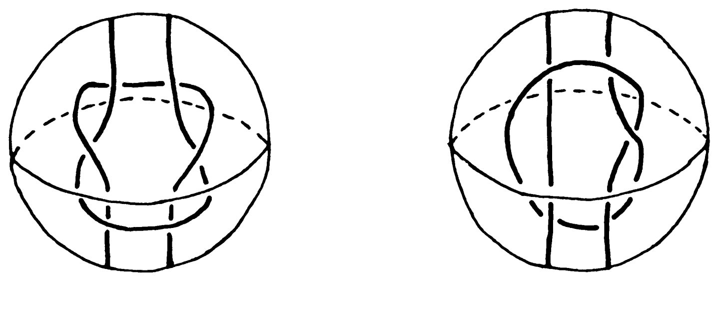
A similar argument applies for $i_-(\pi_1(f(F)))$. In this case the meridian to $K$ is represented by $i_-(\delta)$. Consideration of $i_-(\alpha)$, $i_-(\beta)$, $i_-(\gamma)$, and $i_-(\delta)$ shows that in addition to $l$, each of $lzw$, $x$, $wl$, and $yx$ are trivial in

$$\pi_1(S^1 \times B^2 - F)/\langle i_-(\pi_1(F)), l \rangle.$$

Again it follows that the quotient is trivial.

$f(S^1 \times \partial B^2)$ **is incompressible in** $S^3 - f(J)$. To prove that $f(J)$ has a companion it is necessary to show that $f(S^1 \times \partial B^2)$ is an incompressible torus in $S^3 - f(J)$. As the knot $K$ is nontrivial, it suffices to show that $S^1 \times \partial B^2$ is incompressible in $S^1 \times B^2 - J$.

There are disjoint properly embedded discs $D_1$ and $D_2$ in $S^1 \times B^2$ such that cutting $(S^1 \times B^2, J)$ along $D_1$ and $D_2$ results in tangles $T_1$ and $T_2$ (illustrated in Figure 3).



$T_1$ $T_2$

FIGURE 3

Suppose there is a properly embedded disc $D$ in $S^1 \times B^2$ disjoint from $J$, and with $\partial D$ a meridian to $S^1 \times B^2$. It follows from a standard innermost circle argument that there is such a $D$ with $D \cap (D_1 \cup D_2) = \varnothing$. (Sketch of proof: First arrange that $\partial D \cap (D_1 \cup D_2) = \varnothing$ via an isotopy. If $D \cap (D_1 \cup D_2)$ contains any trivial circles on $(D_1 \cup D_2) - J$, swap a disc on $D$ with an innermost such disc on $(D_1 \cup D_2)$. If after repeating this procedure to eliminate all such circles of intersection any circles of intersection remain, an innermost disc on $D$ along with an annulus from one of the $D_i$ will form a new disc $D'$ with the desired properties.)

Hence, if $S^1 \times \partial B^2$ is compressible in $S^1 \times B^2 - J$, the horizontal equator bounds an embedded disc in either $B^3 - T_1$ or $B^3 - T_2$. This is impossible for $T_1$ by linking number arguments. Kirby and Lickorish prove it is impossible for $T_2$ in [**KL**].

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