VERTICES OF LOCALIZED IMBALANCE
IN A BIASED GRAPH

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ABSTRACT. A *biased graph* consists of a graph Γ and a subclass $\mathcal{B}$ of the polygons of Γ, such that no theta subgraph of Γ contains exactly two members of $\mathcal{B}$. A subgraph is *balanced* when all its polygons belong to $\mathcal{B}$. A vertex is a *balancing vertex* if deleting it leaves a balanced graph. We give a construction for unbalanced biased graphs having a balancing vertex and we show that an unbalanced biased graph having more than one balancing vertex is an unbalanced series or parallel connection of balanced graphs.

Introduction. A *polygon* in a graph Γ is the edge set of a simple closed path. A subclass of polygons of Γ is a *linear subclass* if no theta subgraph of Γ contains exactly two polygons in the subclass. A *biased graph* consists of a graph Γ and a linear subclass $\mathcal{B}$ of polygons of Γ [**2**]. A subgraph or edge set is *balanced* if every polygon in it belongs to $\mathcal{B}$. Biased graphs are a generalization of ordinary graphs, in the sense that for most purposes the latter can be treated as biased graphs that are balanced. For instance, a biased graph $(\Gamma, \mathcal{B})$ has a "bias matroid" $G(\Gamma, \mathcal{B})$ and a "lift matroid" $L(\Gamma, \mathcal{B})$ (see [**3**]), which when $(\Gamma, \mathcal{B})$ is balanced equal the usual polygon matroid of Γ.

We say that the imbalance of a biased graph $(\Gamma, \mathcal{B})$ is *localized at a vertex* v if the vertex-deleted subgraph $\Gamma \setminus v$ is balanced. Briefly, we call v a *balancing vertex*. We are interested in biased graphs that have balancing vertices because in several senses they are among the least unbalanced. For one thing, by definition they are the unbalanced graphs in which balance results by deleting the fewest vertices. They are also the simplest kind of unbalanced graph having no two vertex-disjoint unbalanced polygons. Finally, they are among those biased graphs whose bias and lift matroids are graphic matroids.

In this article we see that the set V^* of balancing vertices of $(\Gamma, \mathcal{B})$ is usually empty and we describe the exceptions (see the Theorem and Corollary 2).

Definitions and a lemma. We let Γ denote a graph (V, E) with vertex set V and edge set E. All graphs are finite. Loops and multiple edges are allowed. By mK_2 we mean m links (edges with two distinct endpoints) in parallel on two vertices. By the *polygon graph* C_l we mean the graph of a simple, closed path of length l; in particular, $C_2 = 2K_2$.

We let $\Phi = (\Gamma, \mathcal{B})$ denote a biased graph whose underlying graph is Γ. We call it *contrabalanced* if $\mathcal{B} = \varnothing$. The *biased union* $\Phi_1 \sqcup \Phi_2$ of biased graphs is their edge-disjoint graph union $(V_1 \cup V_2, E_1 \dot\cup E_2)$ with the balanced polygon class $\mathcal{B}_1 \cup \mathcal{B}_2$. The *contraction* Φ/Δ of Φ by a balanced and connected subgraph Δ is the usual contraction graph Γ/Δ (where $V(\Delta)$ becomes a vertex and the edges of Δ are discarded) with the balanced polygon class

$$\{C \in \mathcal{B} : C \subseteq E \setminus E(\Delta) \text{ and } C \text{ is a polygon in } \Gamma/\Delta\}$$

$$\cup \{C \setminus E(\Delta) : C \in \mathcal{B} \text{ and } C \cap E(\Delta) \text{ is a simple open path}\}.$$

(This definition is drawn from [**2**], where arbitrary contractions are defined.)

In Γ the *bridges* of a vertex set $U \subseteq V$ are the maximal connected subgraphs Δ meeting U and such that Δ is connected through $V \setminus U$; we count an edge with endpoints in U as a bridge [**1**]. We call Δ a *k-bridge* if it contains at least k vertices of U. A *bridge of a vertex* v is merely a bridge of $\{v\}$. A *cutpoint* is a vertex with more than one bridge. A *block of* Γ is a maximal subgraph with no cutpoint. For instance, a loop is a block. A *cutpair* is a pair of vertices lying in a common block and having in that block at least two bridges. (This is nonstandard. Usually one requires some sort of nontriviality.) We apply the same terminology to a biased graph, ignoring its bias.

Let Γ and Γ_1 be two graphs and let $e \in E$, $e_1 \in E_1$ be links; we require that $|E|, |E_1| \geq 2$. The *2-sum* of Γ and Γ_1 at e and e_1 is obtained in the following way. We choose isomorphic copies of Γ and Γ_1, if necessary, to assure that the vertices of e and e_1 are the same but Γ and Γ_1 have nothing else in common. The 2-sum is the union $\Gamma \cup \Gamma_1$ with e and e_1 deleted. We may say Γ_1 is *2-summed into* Γ *at* $e_1 \in E_1$ (and at $e \in E$).

Suppose $\Phi = (\Gamma, \mathcal{B})$ and $\Phi_1 = (\Gamma_1, \mathcal{B}_1)$ are biased graphs. Their *2-sum* at $e_1 \in E_1$ and $e \in E$ is the 2-sum of Γ and Γ_1 with the balanced polygon class

$\mathcal{B} \cup \mathcal{B}_1 \cup \{P \cup P_1 \subseteq (E \cup E_1) \setminus \{e, e_1\} : P \cup P_1 \text{ is a polygon and}$

$\qquad\qquad\qquad\qquad P \cup e \text{ and } P_1 \cup e_1 \text{ are balanced polygons}\}.$

Let us say two polygons have *equal* (or *similar*) *balance* (or other synonyms) if both are balanced or both are unbalanced.

LEMMA 1. *Let Φ be a biased graph and let $u, v \in V(\Phi)$ be a cutpair whose bridges are balanced. Let P_i, Q_i be simple paths from u to v in the ith bridge. Then for each $i \neq j$, $P_i \cup P_j$ and $Q_i \cup Q_j$ have equal balance.*

PROOF. We may take $P_j = Q_j$, for this case implies the lemma. For fixed P_i we argue by descent on the length $l(Q_i)$ of the initial segment common to P_i and Q_i in travelling from u to v. We may assume $P_i \neq Q_i$.

Let x be the first vertex after which P_i and Q_i diverge and let y be the first vertex at which P_i again meets Q_i. Let R_i consist of the path P_i from u to y and Q_i from then on to v. Clearly, $l(R_i) > l(Q_i)$, so inductively $P_i \cup P_j$ and $R_i \cup P_j$ have similar balance. Since $Q_i \cup R_i \cup P_j$ is a theta graph in which the polygon formed in $Q_i \cup R_i$ is balanced, the two remaining polygons, that is $Q_i \cup P_j$ and $R_i \cup P_j$, are equally balanced. The lemma follows. □

We remark that for gain graphs [**2**] (called "voltage graphs" in [**5**]) Lemma 1 has a one-line proof based on potential functions. But to treat biased graphs in general it seems to be necessary to use the stepwise "homotopy" method of our proof.

The structure at a balancing vertex. We can give a construction for biased graphs whose imbalance is localized at a vertex.

PROPOSITION 1. *Let Φ be a biased graph and v a vertex. The vertex-deleted subgraph $\Phi \setminus v$ is balanced if and only if Φ has the form $(\Psi \sqcup T)/T$, where Ψ and T are balanced biased graphs, T is connected, and $V(T) \subseteq V(\Psi)$. (Then v in Φ corresponds to the vertex to which T is contracted in $(\Psi \sqcup T)/T$.)*

We note that it alway suffices to let T be a tree.

PROOF. First, we note that, if Φ has the stated form, then $\Phi \setminus v = \Psi \setminus V(T)$, which is balanced.

For the converse, let Φ be biased and v be a vertex such that $\Omega = \Phi \setminus v$ is balanced. Let $\Omega_1, \ldots, \Omega_r$ be the connected components of Ω and let E_i be the set of links from v to Ω_i. If $e \in E_i$, let w_e be its vertex other than v. We define an equivalence relation $\sim$ on each set E_i by $e \sim f$ if $\Omega_i \cup (V, \{e, f\})$ is a balanced subgraph of Φ. To prove this is an equivalence relation, suppose $e \sim f \sim g$ in E_i. Let P be a simple path from e to g in Ω_i and let Q be a simple path from f to P in Ω_i such that $P \cup Q \cup \{e, f, g\}$ is a theta graph. Let P_e be the part of P from e to Q and P_g the part from Q to g. By hypothesis $P_e \cup Q \cup \{e, f\}$ and $P_g \cup Q \cup \{f, g\}$ are balanced; consequently, $P \cup \{e, g\}$ is balanced. We conclude that $e \sim g$. Let $E_{i1}, E_{i2}, \ldots$ be the equivalence classes of E_i.

Let m_i be the number of equivalence classes of E_i and let $m_0 = \max m_i$. Let $m = m_0$, except that $m = 2$ if $m_0 = 1$ and there is an unbalanced loop at v. We define Ψ to be the balanced graph with vertex set $(V \setminus \{v\}) \cup U$, where $U = \{u_1, \ldots, u_m\}$ is disjoint from V, and edge set E connected as follows: Ψ restricted to $V \setminus \{v\}$ equals $\Phi \setminus v$; an edge $e \in E_{ij}$ is incident to u_j instead of v, a balanced loop at v becomes one at u_1, and an unbalanced loop at v becomes a link from u_1 to u_2. Let T be any tree spanning U and let $\Phi' = (\Psi \sqcup T)/T$.

We must show that $\Phi' = \Phi$; then the proof will be complete. The underlying graphs are clearly the same and the loops at v have the proper amount of balance. Consider a polygon C at v, whose edges at v are e and f. Both e and f must lie in some E_i. If $e \sim f$, then C is balanced in Φ so e and f, in Ψ, are connected to the same vertex u_j. Then C remains a balanced polygon in Ψ and therefore remains balanced in Φ'. But if $e \nsim f$, then e and f are connected in Ψ to different vertices, say u_j and u_k respectively. In Ψ, C is an open path, completed to a polygon D by the $u_j u_k$-path in T. Since D is unbalanced, C becomes, in Φ', an unbalanced polygon. We are done, therefore, if we show that C was unbalanced in Φ; or, it suffices to say, in its subgraph $\Omega_i \cup (V, \{e, f\})$. In the latter graph, $\{w_e, w_f\}$ is a cutpair and Ω_i and the graph $(\{v, w_e, w_f\}, \{e, f\})$ are its bridges. Both bridges are balanced; thus by Lemma 1, since $\Omega_i \cup (V, \{e, f\})$ is not balanced, C is unbalanced in Φ. $\square$

It is easy to see that m is the smallest number of vertices that can be given to T in the proposition.

A biased graph Φ is *sign-biased* if there is a sign labelling $\sigma: E \to \{+, -\}$ such that a polygon is in $\mathcal{B}$ precisely when the product of its edge signs is $+$. By [4,

Theorem 6], a necessary and sufficient condition for Φ to be sign-biased is that no theta graph have all its polygons unbalanced.

COROLLARY 1. *If Φ is sign-biased and $\Phi \setminus v$ is balanced, then $|V(T)|$ can be taken to be 2 (or 1 if Φ is balanced).*

PROOF. Suppose in Ω_1, say, there were three mutually inequivalent edges e_1, e_2, e_3. Let P be a path in Ω_1 from e_1 to e_2 and let Q be a path in Ω_1 from e_3 to P. Then $P \cup Q \cup \{e_1, e_2, e_3\}$ is a theta graph in which no polygon is balanced. This contradicts the cited result of [**4**]. □

We supplement this corollary with a converse.

PROPOSITION 2. *Suppose Φ is a biased graph of the form $(\Psi \sqcup T)/T$, where Ψ and T are as in Proposition 1. Then Φ is sign-biased if and only if $V(T)$ has no 3-bridge in Ψ.*

PROOF. Let v be the vertex in Φ corresponding to $V(T)$ in Ψ. Suppose there is no 3-bridge. Consider a theta graph in Φ composed of paths P_1, P_2, P_3 meeting at vertices x and y. If $x, y \neq v$, the theta graph contains a balanced polygon composed of two paths not meeting v. If $y = v$, say, then in Ψ at least two of the paths, say P_1 and P_2, end at the same vertex $t \in V(T)$. Then $P_1 \cup P_2$ is balanced.

If there is a 3-bridge, it contains three internally disjoint paths from some vertex $x \notin V(T)$ to distinct vertices $t_1, t_2, t_3 \in V(T)$. In Φ these paths form a theta graph having no balanced polygon. □

One more remark about the case of Corollary 1. Let $G(\Phi)$ be the bias matroid defined in [**3**] (for sign-biased Φ also see [**5**]). If $\Phi = (\Psi \sqcup T)/T$, where Ψ is balanced and T is an edge, then not only is Φ sign-biased, but even more $G(\Phi)$ is graphic, indeed isomorphic to the polygon matroid of Ψ. (For the proof see [**6**, Lemma 1F].)

The set of balancing vertices. Suppose Ψ is a graph and $U \subseteq V(T)$. Let $W_U(\Psi)$ be the set of cutpoints $w \notin U$ such that w separates all the members of U in Ψ. (We are assuming $|U| \geq 2$.) We can now state our theorem.

THEOREM. *Let Φ be a biased graph and V^* its set of balancing vertices. Then $V^* = \varnothing$ except in the following cases:*

(i) *When Φ is balanced. Then $V^* = V$.*

(ii) *When Φ is unbalanced and has the form $(\Psi \sqcup T)/T$ where Ψ and T are balanced, T is connected, and $V(T) \subseteq V(\Psi)$. Let v be the vertex in Φ corresponding to $V(T)$. Supposing $V(T)$ has at least two 2-bridges in Ψ, then Φ has more than one unbalanced block and $V^* = \{v\}$. Supposing $V(T)$ has exactly one 2-bridge in Ψ, then Φ has just one unbalanced block and $V^* = \{v\} \cup W_{V(T)}(\Psi)$.*

PROOF. Part (i) is obvious.

In part (ii) it is clear that each unbalanced block of $\Phi = (\Psi \sqcup T)/T$ arises from a 2-bridge, and vice versa. Note that $|V(T)| \geq 2$ because Φ is not balanced.

If $V(T)$ has at least two 2-bridges, then the only point common to every unbalanced block in Φ is v. Clearly, $\Phi \setminus v$ is balanced. Therefore, $V^* = \{v\}$.

If $V(T)$ has only one 2-bridge, again $v \in V^*$ but there may be other elements. For w to belong to V^* it is necessary and sufficient that in $\Psi \setminus w$ there is no path connecting two vertices of T. Thus, we have proved Φ behaves as described in part (ii).

Now suppose Φ is an arbitrary unbalanced biased graph with $V_1(\Phi) \neq \varnothing$. Let $v \in V^*$. Then $\Phi = (\Psi \sqcup T)/T$ as described in part (ii), by Proposition 1. $V(T)$ must have at least one 2-bridge in Ψ, or else Φ would have been balanced. Thus we have $V^* \neq \varnothing$ precisely when the theorem says. $\square$

COROLLARY 2. *For a biased graph Φ we have $V^* = \varnothing$ except in the following cases*:

(i) *If Φ is balanced, $V^* = V$.*

(ii) *If Φ has exactly one unbalanced block, which is formed by 2-summing anywhere from 0 to m balanced graphs into a contrabalanced graph $(mK_2, \varnothing)$ where $m \geq 3$, then $V^* = V(mK_2)$.*

(iii) *If Φ has exactly one unbalanced block, which is formed by 2-summing any number from 0 to l of balanced graphs Φ_i (summed on the edge $e_i \in \Phi_i$) into an unbalanced polygon graph $(C_l, \varnothing)$ where $l \geq 2$, and if in each $\Phi_i \setminus e_i$ there is no cutpoint separating the endpoints of e_i, then $V^* = V(C_l)$.*

(iv) *If Φ has the form $(\Psi \sqcup T)/T$ as in Proposition 1 and Φ is not as described in (i), (ii), or (iii), then $|V^*| = 1$.*

A biased graph Φ constructed as in (ii) can be thought of as an *unbalanced parallel connection* of m balanced graphs, some of which may be single edges. A Φ constructed as in (iii) can be regarded as an *unbalanced series connection* of l balanced graphs, some of which may be single edges.

PROOF. The difference from the Theorem is in parts (ii) and (iii). We have to determine just when in part (ii) of the Theorem there is a point in V^* besides v. Thus, $V(T)$ must have exactly one 2-bridge in Ψ and $W_{V(T)}(\Psi)$ should be nonempty. We may as well assume that Ψ is connected. Notice that $V(T)$ is stable in Ψ.

If $V(T) = \{u_1, \ldots, u_m\}$, $m \geq 3$, there can be only one point w in $W_{V(T)}(\Psi)$ if it is not empty. Let the bridges of $\{w, u_1, \ldots, u_m\}$ be Ψ_{ij} for $i = 1, \ldots, m$, where $u_i \in V(\Psi_{ij})$, and Ψ_j, where no u_i is a vertex of Ψ_j. Let Φ_1 be the union of all Ψ_{1j} and Ψ_j and for $i \geq 2$ let Φ_i be the union of all Ψ_{ij}, with an additional link e_i, whose endpoints are w and u_i, added to each Φ_i for $i = 1, \ldots, m$. It is clear that Φ is obtained by 2-summing each Φ_i at e_i to a different edge of an $(mK_2, \varnothing)$ whose vertices are w and v.

Suppose that $V(T) = \{u_1, u_2\}$. Then $W_{V(T)}(\Psi)$ can be arbitrarily large; say it equals $\{w_2, w_3, \ldots, w_l\}$ where $l \geq 2$, and let $w_1 = v$ (corresponding in Φ to $\{u_1, u_2\}$ in Ψ). Evidently Ψ has the form of a path P whose vertices are, in order, $u_1, w_2, w_3, \ldots, w_l, u_2$, modified by 2-summing into each edge e_i' (where, say, $e_1' = u_1 w_2$, $e_i' = w_i w_{i+1}$ for $2 \leq i < l$, and $e_l' = w_l u_2$) a suitable balanced graph Φ_i along its edge e_i. In order for the 2-sum to introduce no additional cutpoints separating u_1 from u_2 in Ψ, we must require that $\Phi_i \setminus e_i$ have no vertex separating the endpoints of e_i. Now since T is a single edge and $(P \sqcup T)/T = (C_l, \varnothing)$ (as a biased graph), we can conclude that Φ is constructed as in (iii). $\square$

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