CHEBYSHEV TYPE ESTIMATES FOR BEURLING GENERALIZED PRIME NUMBERS

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ABSTRACT. We consider a Beurling generalized prime system for which the distribution function $N(x)$ of the integers satisfies

$$\int_1^\infty x^{-1} \left\{ \sup_{x \leq y} \frac{|N(y) - Ay|}{y} \right\} dx < \infty$$

with constant $A > 0$. We shall prove that the Chebyshev type estimates

$$0 < \liminf_{x \rightarrow \infty} \frac{\psi(x)}{x}, \quad \limsup_{x \rightarrow \infty} \frac{\psi(x)}{x} < \infty$$

hold for the system. This gives a partial proof of one of Diamond's conjectures.

Chebyshev was the first to establish the correct order of magnitude of the weighted counting function $\psi(x)$ of the ordinary prime numbers. He showed that there exist two numbers $\alpha > 0$ and $\beta < \infty$ such that

$$(1) \quad \liminf_{x \rightarrow \infty} \frac{\psi(x)}{x} \geq \alpha, \quad \limsup_{x \rightarrow \infty} \frac{\psi(x)}{x} \leq \beta.$$

The prime number theorem (P.N.T.) asserts that $\alpha = \beta = 1$. Here we shall study Chebyshev type estimates for Beurling generalized primes.

Let $\mathcal{P} = \{p_i\}_{i=1}^\infty$, where $1 < p_1 \leq p_2 \leq \dots$, $p_i \rightarrow \infty$, be a set of Beurling generalized (henceforth g -) prime numbers and $\mathcal{N} = \{n_i\}_{i=0}^\infty$ be the associated set of g -integers (see [1, 2]). Define

$$N(x) = \sum_{\substack{i \\ n_i \leq x}} 1, \quad \psi(x) = \sum_{\substack{i, \alpha \\ p_i^\alpha \leq x}} \log p_i.$$

Beurling [2] proved that if

$$(2) \quad N(x) = Ax + O(x \log^{-\gamma} x)$$

for some constants $A > 0$ and $\gamma > 3/2$, then the P.N.T. holds for $\mathcal{P}$. If $\gamma = 3/2$ in (2), the P.N.T. need not hold as Diamond [3] showed by an example based on a continuous example of Beurling. Diamond [4] also showed that if $\gamma > 1$ in (2) then (1) holds. On the other hand, (1) is not generally true if $\gamma < 1$ in (2), as an example

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of Hall [6] shows. Moreover, Diamond [5] conjectured that the weaker condition

$$\int_1^{\infty} x^{-2} |N(x) - Ax| dx < \infty$$

is sufficient to imply Chebyshev type bounds for a g -prime system.

In the present paper, we shall prove the following

THEOREM. *If the distribution function $N(x)$ of the g -integers of a g -prime system satisfies*

$$(3) \quad \int_1^{\infty} x^{-1} \left\{ \sup_{x \leq y} \frac{|N(y) - Ay|}{y} \right\} dx < \infty$$

with constant $A > 0$, then there exist numbers $\alpha > 0$ and $\beta < \infty$ for which (1) holds.

This theorem gives a partial proof of Diamond's conjecture.

We divide the proof of the Theorem into several lemmas.

LEMMA 1. *Assume (3). Then there exists a function $Q(x)$ such that*

$$(4) \quad Q(x) \text{ is nonincreasing,}$$

$$(5) \quad \int_1^{\infty} Q(x) x^{-1} dx < \infty,$$

$$(6) \quad Q(x) \leq 4Q(x^2) \quad \text{for all } x \geq 1,$$

and

$$(7) \quad Q(x) \geq \sup_{x \leq y} \frac{|N(y) - Ay|}{y}.$$

PROOF. Let

$$Q_1(x) = \sup_{x \leq y} \frac{|N(y) - Ay|}{y}.$$

We note that $Q_1(x)$ is nonnegative and nonincreasing. Define $Q(x)$ recursively by setting

$$Q(x) = \begin{cases} Q_1(1) & \text{for } 1 \leq x \leq 2, \\ \max\{Q_1(2^{2^{m-1}}), 4^{-1}Q(2^{2^{m-1}})\} & \text{for } 2^{2^{m-1}} < x \leq 2^{2^m}, m \in \mathbf{N}. \end{cases}$$

We now verify that this function satisfies each of the following conditions:

(i) $Q(x) \geq Q_1(x)$ (this is obvious).

(ii) $Q(x) \downarrow$.

We note that $Q(x)$ is constant on $2^{2^{m-1}} < x \leq 2^{2^m}$. If $1 \leq x_1 < 2 < x_2 \leq 2^2$, then

$$Q(x_2) = \max\{Q_1(2), 4^{-1}Q(2)\} \leq Q_1(1) = Q(x_1).$$

If $2^{2^{m-1}} < x_1 \leq 2^{2^m} < x_2 \leq 2^{2^{m+1}}$, then

$$Q(x_2) = \max\{Q_1(2^{2^m}), 4^{-1}Q(2^{2^m})\} \leq Q(2^{2^m}) = Q(x_1)$$

since, by (i), $Q_1(2^{2^m}) \leq Q(2^{2^m})$.

(iii) $Q(x) \leq 4Q(x^2)$.

For $1 \leq x \leq 2$, we have $1 \leq x^2 \leq 2^2$. If $1 \leq x^2 \leq 2$, we have nothing to show. If $2 < x^2 \leq 2^2$, we have

$$Q(x^2) \geq 4^{-1}Q(2) = 4^{-1}Q(x).$$

For $2^{2^{m-1}} < x \leq 2^{2^m}$, $m \geq 1$, we have $2^{2^m} < x^2 \leq 2^{2^{m+1}}$ and hence

$$Q(x^2) \geq 4^{-1}Q(2^{2^m}) = 4^{-1}Q(x).$$

(iv) $\int_1^\infty Q(x)x^{-1}dx < \infty$.

We have to show that

$$\sum_{m=1}^{\infty} \int_{2^{2^{m-1}}}^{2^{2^m}} Q(x)x^{-1}dx < \infty.$$

There are two distinct cases that we have to consider separately.

Case I. $4^{-1}Q(2^{2^m}) \geq Q_1(2^{2^m})$ for all $m \geq m_0$. In this case, for $2^{2^{m_0}} < x \leq 2^{2^{m_0+1}}$,

$$Q(x) = \max\{Q_1(2^{2^{m_0}}), 4^{-1}Q(2^{2^{m_0}})\} = 4^{-1}Q(2^{2^{m_0}}).$$

By induction, for all $m \geq m_0 + 1$, $2^{2^{m-1}} < x \leq 2^{2^m}$, we have

$$Q(x) = 4^{-1}Q(2^{2^m}) = 4^{-(m-m_0)}Q(2^{2^{m_0}}).$$

Therefore,

$$\begin{aligned} \sum_{m=m_0+1}^{\infty} \int_{2^{2^{m-1}}}^{2^{2^m}} Q(x)x^{-1}dx &= \sum_{m=m_0+1}^{\infty} 4^{-(m-m_0)}Q(2^{2^{m_0}}) \int_{2^{2^{m-1}}}^{2^{2^m}} x^{-1}dx \\ &= \sum_{m=m_0+1}^{\infty} 4^{-(m-m_0)}Q(2^{2^{m_0}}) \log(2^{2^{m-1}}) = Q(2^{2^{m_0}}) \log(2^{2^{m_0-1}}). \end{aligned}$$

Case II. There exist $m_1 < m_2 < \dots < m_k < m_{k+1} < \dots$ such that $Q_1(2^{2^{m_k}}) > 4^{-1}Q(2^{2^{m_k}})$ and $Q_1(2^{2^m}) \leq 4^{-1}Q(2^{2^m}) \forall m \neq m_k$. In this case, we can show that

$$\sum_{m=m_k+1}^{m_{k+1}} \int_{2^{2^{m-1}}}^{2^{2^m}} Q(x)x^{-1}dx \leq 4Q_1(2^{2^{m_k}}) \log(2^{2^{m_k-1}}).$$

Actually, if $m_{k+1} = m_k + 1$, the left-hand side equals

$$\begin{aligned} \int_{2^{2^{m_k}}}^{2^{2^{m_k+1}}} Q(x)x^{-1}dx &= Q_1(2^{2^{m_k}}) \int_{2^{2^{m_k}}}^{2^{2^{m_k+1}}} x^{-1}dx \\ &= Q_1(2^{2^{m_k}}) \log(2^{2^{m_k}}) = 2Q_1(2^{2^{m_k}}) \log(2^{2^{m_k-1}}). \end{aligned}$$

Therefore, we consider $m_k < m_k + 2 \leq m_{k+1}$. For $2^{2^{m_k}} < x \leq 2^{2^{m_k+1}}$, we have

$$Q(x) = \max\{Q_1(2^{2^{m_k}}), 4^{-1}Q(2^{2^{m_k}})\} = Q_1(2^{2^{m_k}}).$$

Then, for $2^{2^{m_k+1}} < x \leq 2^{2^{m_k+2}}$,

$$Q(x) = 4^{-1}Q(2^{2^{m_k+1}}) = 4^{-1}Q_1(2^{2^{m_k}}).$$

By induction, for all $m \in [m_k + 1, m_{k+1}]$ and $2^{2^{m-1}} < x \leq 2^{2^m}$, we have $Q(x) = 4^{-(m-m_k)+1} Q_1(2^{2^{m_k}})$. Therefore,

$$\begin{aligned} \sum_{m=m_k+1}^{m_{k+1}} \int_{2^{2^{m-1}}}^{2^{2^m}} Q(x) x^{-1} dx &= \sum_{m=m_k+1}^{m_{k+1}} 4^{-(m-m_k)+1} Q_1(2^{2^{m_k}}) \int_{2^{2^{m-1}}}^{2^{2^m}} x^{-1} dx \\ &= \sum_{m=m_k+1}^{m_{k+1}} 4^{-(m-m_k)+1} Q_1(2^{2^{m_k}}) \log(2^{2^{m-1}}) \\ &= Q_1(2^{2^{m_k}}) \log(2^{2^{m_k-1}}) 4 \sum_{k=1}^{m_{k+1}-m_k} 2^{-k} \leq 4 Q_1(2^{2^{m_k}}) \log(2^{2^{m_k-1}}). \end{aligned}$$

Thus we have

$$\begin{aligned} \sum_{m=m_1+1}^{\infty} \int_{2^{2^{m-1}}}^{2^{2^m}} Q(x) x^{-1} dx &= \sum_{k=1}^{\infty} \sum_{m=m_k+1}^{m_{k+1}} \int_{2^{2^{m-1}}}^{2^{2^m}} Q(x) x^{-1} dx \\ &\leq 4 \sum_{k=1}^{\infty} Q_1(2^{2^{m_k}}) \log(2^{2^{m_k-1}}) \leq 4 \sum_{m=1}^{\infty} Q_1(2^{2^m}) \log(2^{2^{m-1}}). \end{aligned}$$

To show the last sum is finite, we note that

$$\int_{2^{2^{m-1}}}^{2^{2^m}} Q_1(x) x^{-1} dx \geq Q_1(2^{2^m}) \int_{2^{2^{m-1}}}^{2^{2^m}} x^{-1} dx = Q_1(2^{2^m}) \log(2^{2^{m-1}})$$

since $Q_1(x) \downarrow$. Therefore, we have

$$\sum_{m=1}^{\infty} Q_1(2^{2^m}) \log(2^{2^{m-1}}) \leq \sum_{m=1}^{\infty} \int_{2^{2^{m-1}}}^{2^{2^m}} Q_1(x) x^{-1} dx = \int_2^{\infty} Q_1(x) x^{-1} dx < \infty$$

by (3). This completes the proof of Lemma 1. $\square$

LEMMA 2. Assume (4) and (5). Then $Q(x) = o(\log^{-1} ex)$.

PROOF. Given $\varepsilon > 0$, for $x \geq x_0$, we have $\int_x^{\infty} Q(t) t^{-1} dt < \varepsilon/2$. Thus, for $x \geq x_0^2$,

$$Q(x) \log \sqrt{x} \leq \int_{\sqrt{x}}^x Q(t) t^{-1} dt < \varepsilon/2,$$

i.e., $Q(x) \log x < \varepsilon$. $\square$

The following lemma shows that $Q(x) dx$ has a kind of “stability” under multiplicative convolution.

LEMMA 3. Assume (4), (5), and (6). Then $\int_1^x t^{-1} Q(x/t) Q(t) dt \leq c_1 Q(x)$, where c_1 is a constant.

PROOF. We have

$$\begin{aligned} \int_1^x t^{-1} Q(x/t) Q(t) dt &= 2 \int_1^{\sqrt{x}} t^{-1} Q(x/t) Q(t) dt \\ &\leq 2 Q(\sqrt{x}) \int_1^{\sqrt{x}} Q(t) t^{-1} dt \leq c_1 Q(x) \end{aligned}$$

since, by (4) and (6), $Q(x/t) \leq Q(\sqrt{x}) \leq 4Q(x)$ for $1 \leq t \leq \sqrt{x}$. $\square$

By using the "stability" of $Q(x) dx$, we can prove the following lemma which is the main step in the proof of the Theorem.

LEMMA 4. Assume (4), (5), (6), and (7). Then, for fixed and sufficiently small $\varepsilon > 0$, we have

$$u_\varepsilon(x) := \int_1^x dN * (\delta - \varepsilon t^{-\varepsilon} dt) * Q(t) dt \geq 0$$

for $x \geq 1$ and $u_\varepsilon(x) \rightarrow \infty$ as $x \rightarrow \infty$.

PROOF. If $1 \leq x \leq 1/\varepsilon$, then

$$\int_1^x dN * (\delta - \varepsilon t^{-\varepsilon} dt) \geq N(x) - \varepsilon N(x)(x-1) \geq 0.$$

The lemma is certainly true for $1 \leq x \leq 1/\varepsilon$, since the third convolution factor is everywhere nonnegative.

If $x > 1/\varepsilon$, we utilize all the convolution factors. We write

$$\begin{aligned} u_\varepsilon(x) &= A \int_1^x (\delta + dt) * (\delta - \varepsilon t^{-\varepsilon} dt) * Q(t) dt \\ &\quad + \int_1^x (dN - A\delta - A dt) * (\delta - \varepsilon t^{-\varepsilon} dt) * Q(t) dt \\ &= I_1 + I_2, \end{aligned}$$

say. We will show that I_1 is positive, that $I_1 \rightarrow \infty$ as $x \rightarrow \infty$ and that I_2 is negligible. Actually, we have

$$I_1 = A \int_1^x (x/t)^{1-\varepsilon} Q(t) dt = Ax \int_1^x u^{-\varepsilon-1} Q(x/u) du$$

since $\int_1^x (\delta + dt) * (\delta - \varepsilon t^{-\varepsilon} dt) = x^{1-\varepsilon}$. It is easy to see that $I_1 \geq Ax^{1-\varepsilon} \int_1^x t^{-1} Q(t) dt \rightarrow \infty$ as $x \rightarrow \infty$. Moreover, by Lemma 3, we have

$$\left| \int_1^x (dN - A\delta - A dt) * Q(t) dt \right| \leq \int_1^x Kxt^{-1} Q(x/t) Q(t) dt \leq c_2 x Q(x).$$

It follows that

$$|I_2| \leq c_2 \int_1^x xt^{-1} Q(x/t) (\delta + \varepsilon t^{-\varepsilon} dt) = c_2 x Q(x) + c_2 \varepsilon x \int_1^x t^{-\varepsilon-1} Q(x/t) dt.$$

For $\varepsilon > 0$ sufficiently small, $c_2 \varepsilon < \frac{1}{3}A$. Also, for $x \geq 1/\varepsilon$, we have

$$\begin{aligned} \frac{1}{2} I_1 &= \frac{1}{2} Ax \int_1^x t^{-\varepsilon-1} Q(x/t) dt \geq \frac{1}{2} Ax Q(x) \int_1^{1/\varepsilon} t^{-\varepsilon-1} dt \\ &\geq \frac{1}{2} Ax Q(x) \varepsilon \log \frac{1}{\varepsilon} \geq c_2 x Q(x) \end{aligned}$$

since $\varepsilon^\varepsilon \geq \exp(-e^{-1})$. Therefore, $u_\varepsilon(x) \geq I_1 - |I_2| \geq \frac{1}{6} I_1$. $\square$

LEMMA 5. Suppose that $N(x) = Ax + O(x \log^{-1} ex)$. Then we have

$$I = \int_1^x L dN * (\delta - \varepsilon t^{-\varepsilon} dt) = O(x).$$

PROOF. We have

$$I = \int_1^x L dN - \int_1^x L dN * \varepsilon t^{-\varepsilon} dt = I_1 - I_2,$$

say. By integration by parts,

$$I_1 = \int_1^x \log t dN(t) = N(x) \log x - \int_1^x N(t) t^{-1} dt = N(x) \log x + O(x).$$

Also, we have

$$\begin{aligned} I_2 &= \varepsilon \int_1^x L dN * t^{-\varepsilon} dt \\ &= \frac{\varepsilon}{1-\varepsilon} \int_1^x \left(\frac{x}{t}\right)^{1-\varepsilon} \log t dN(t) - \frac{\varepsilon}{1-\varepsilon} \int_1^x \log t dN(t) \\ &= \frac{\varepsilon}{1-\varepsilon} I_3 - \frac{\varepsilon}{1-\varepsilon} I_1, \end{aligned}$$

say. Again, by integration by parts,

$$\begin{aligned} I_3 &= \int_1^x x^{1-\varepsilon} t^{-1+\varepsilon} \log t dN(t) \\ &= N(x) \log x - x^{1-\varepsilon} \int_1^x N(t) \{(-1+\varepsilon)t^{-2+\varepsilon} \log t + t^{-2+\varepsilon}\} dt \\ &= N(x) \log x - x^{1-\varepsilon} (I_4 + I_5), \end{aligned}$$

say. It is easy to see that

$$\begin{aligned} I_4 &= (-1+\varepsilon) \int_1^x \{At + O(t \log^{-1} \varepsilon t)\} t^{-2+\varepsilon} \log t dt \\ &= (-1+\varepsilon) \left\{ A \frac{x^\varepsilon}{\varepsilon} \log x + O_\varepsilon(x^\varepsilon) \right\}. \end{aligned}$$

Moreover,

$$I_5 = \int_1^x N(t) t^{-2+\varepsilon} dt = O\left(\int_1^x t^{-1+\varepsilon} dt\right) = O_\varepsilon(x^\varepsilon).$$

Combining all the estimates, we obtain $I = O(x)$. $\square$

LEMMA 6. Assume (4), (5), (6), and (7). Then

$$\int_1^x L dN * (\delta - \varepsilon t^{-\varepsilon} dt) * Q(t) dt = O(x).$$

PROOF. By Lemma 2 and Lemma 5, the integral on the left-hand side equals

$$\int_1^x O(x/t) Q(t) dx = O\left(x \int_1^x t^{-1} Q(t) dt\right) = O(x). \quad \square$$

We are now in the position to set up the upper estimate of the Theorem. The starting-point of the proof is Chebyshev's identity $d\psi * dN = LdN$ which, in this form, is still valid for Beurling's g -prime system. We convolve each side of it by $(\delta - \varepsilon t^{-\varepsilon} dt) * Q(t) dt$ and obtain

$$\int_1^x dN * (\delta - \varepsilon t^{-\varepsilon} dt) * Q(t) dt * d\psi = \int_1^x L dN * (\delta - \varepsilon t^{-\varepsilon} dt) * Q(t) dt.$$

Proof of the upper estimate. Assume (3), then (4), (5), (6), and (7) hold by Lemma 1. By Lemma 4 and Lemma 6,

$$\psi(x/B) \leq \int_1^x u_\epsilon(x/t) d\psi(t) = O(x)$$

since $u_\epsilon(x) \geq 1$ for $x \geq B$. $\square$

To set up the lower estimate of the Theorem, we need one more lemma.

LEMMA 7. Suppose that $N(x) = Ax + o(x \log^{-1} ex)$. Then

$$I = \int_1^x L dN * (\delta - t^{-1} dt) = Ax + o(x).$$

PROOF. We have $I = \int_1^x L dN - \int_1^x L dN * t^{-1} dt = I_1 - I_2$, say. By integration by parts,

$$\begin{aligned} I_1 &= \int_1^x \log t dN(t) = N(x) \log x - \int_1^x N(t) t^{-1} dt \\ &= N(x) \log x - Ax + o(x \log^{-1} ex). \end{aligned}$$

Moreover, we have

$$\begin{aligned} I_2 &= \int_1^x \log(x/t) \log t dN(t) \\ &= \log x \int_1^x \log t dN(t) - \int_1^x \log^2 t dN(t) = (\log x) I_1 - I_3, \end{aligned}$$

say. Similarly, by integration by parts, $I_3 = N(x) \log^2 x - 2Ax(\log x - 1) + o(x)$. Combining the results, we arrive at

$$I = (1 - \log x) I_1 + I_3 = Ax + o(x). \quad \square$$

Proof of the lower estimate. Assume (3), then (4), (5), (6), and (7) hold by Lemma 1. We have

$$\begin{aligned} u_1(x) &:= \int_1^x dN * (\delta - t^{-1} dt) = N(x) - \int_1^x N(x/t) t^{-1} dt \\ &= N(x) - \int_1^x \{ Axt^{-1} + O(xt^{-1}Q(x/t)) \} t^{-1} dt \\ &= N(x) - Ax + A + O\left(\int_1^x xt^{-2}Q(x/t) dt\right) \\ &= O\left(1 + \int_1^x Q(t) dt\right) \end{aligned}$$

since $|N(x) - Ax| \leq xQ(x) \leq 2 \int_1^x Q(t) dt$ for $x \geq 2$. From Chebyshev's identity, we have

$$\int_1^x dN * (\delta - t^{-1} dt) * d\psi = \int_1^x L dN * (\delta - t^{-1} dt).$$

By Lemma 2 and Lemma 7, the integral on the right-hand side is $Ax + o(x)$. Therefore, we have $\int_1^x u_1(x/t) d\psi(t) = Ax + o(x)$. We note that, by the upper estimate,

$$\begin{aligned} \int_1^x u_1(x/t) d\psi(t) &\leq K \int_1^x \left(1 + \int_1^{x/t} Q(u) du\right) d\psi(t) \\ &\leq K\psi(x) + K \int_1^x d\psi * Q(t) dt = K\psi(x) + K \int_1^x \psi(x/t) Q(t) dt \\ &\leq K \left\{ \psi(x) \left(1 + \int_1^B Q(t) dt\right) + c_3 x \int_B^x t^{-1} Q(t) dt \right\} \\ &\leq c_4 \psi(x) + c_3 Kx \int_B^\infty t^{-1} Q(t) dt. \end{aligned}$$

For B sufficiently large, $c_3 K \int_B^\infty t^{-1} Q(t) dt \leq \frac{1}{3}A$. Fixing B , for x sufficiently large, we have $c_4 \psi(x) \geq \frac{1}{3}Ax$. This completes the proof of the Theorem. $\square$

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