NONISOMORPHIC ALGEBRAIC STRUCTURES ON SMOOTH MANIFOLDS

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ABSTRACT. Given a compact smooth hypersurface M in $\mathbf{R}^{n+1}$, we construct a family $\{X_k\}$, $k = 1, 2, \dots$, of nonsingular real algebraic subsets of $\mathbf{R}^{n+1}$ such that each X_k is isotopic to M but, for $k \neq l$, X_k and X_l are not birationally equivalent.

Let $X \subset \mathbf{R}^n$ and $Y \subset \mathbf{R}^p$ be locally closed sets in the Zariski topology. A map $f: X \rightarrow Y$ is said to be regular if there exist polynomials f_i, g_i in $\mathbf{R}[x_1, \dots, x_n]$, $g_i^{-1}(0) \cap X = \emptyset$, $i = 1, \dots, p$, such that $f(x) = (f_1(x)/g_1(x), \dots, f_p(x)/g_p(x))$ for all x in X . A bijective map $f: X \rightarrow Y$ is called a regular isomorphism if f and f^{-1} are regular maps; X and Y are called birationally equivalent if there exist Zariski open and dense subsets U and V of X and Y , respectively, which are regularly isomorphic.

Let M be a compact smooth (i.e., C^∞) manifold and let $\Omega_{\text{alg}}(M)$ be the set of all equivalence classes of pairs (X, φ) , where X is a nonsingular real algebraic set, $\varphi: X \rightarrow M$ is a smooth diffeomorphism, and two pairs (X_1, φ_1) , (X_2, φ_2) are considered to be equivalent if there exists a regular isomorphism $\gamma: X_1 \rightarrow X_2$ satisfying $\varphi_1 = \varphi_2 \circ \gamma$. Thus $\Omega_{\text{alg}}(M)$, originally introduced by Akbulut and King [1], is the set of distinct algebraic structures on M (the *moduli space* of M).

It is well known that if we replace in the above definition of $\Omega_{\text{alg}}(M)$ the real data by the complex ones, i.e., if M is a compact complex manifold, X a complex projective algebraic set, φ a biholomorphism, and γ a complex regular isomorphism, then the corresponding space $\Omega_{\text{alg}}(M, \mathbf{C})$ is either empty or has exactly one element (cf. [5, p. 68]). The situation is quite different in the real case. The space $\Omega_{\text{alg}}(M)$ is never empty by a well-known theorem of Nash-Tognoli [1]. Akbulut and King [1, 2] described how to construct smooth manifolds M with $\Omega_{\text{alg}}(M)$ containing at least two elements, but their method does not allow us to prove that $\Omega_{\text{alg}}(M)$ is infinite. On the other hand one easily shows that $\Omega_{\text{alg}}(S^n)$, where S^n is the unit n -dimensional sphere, is an infinite set provided $n \geq 1$. Indeed, for each integer k , $k \geq (n+1)/2$, consider the complex algebraic projective set $V_k = \{[z_1, \dots, z_{n+2}] \in \mathbf{CP}^{n+1} \mid z_1^{2k} + \dots + z_{n+1}^{2k} = z_{n+2}^{2k}\}$. Clearly, V_k is nonsingular and its arithmetic genus $p_a(V_k)$ is equal to $\binom{2k-1}{n+1}$ [4, p. 54]. Thus if $k \neq l$, then $p_a(V_k) \neq p_a(V_l)$ and hence V_k and V_l are not birationally equivalent as complex varieties (cf. [4, p. 431]). Now identify $\mathbf{R}^{n+1}$ with a subset of $\mathbf{CP}^{n+1}$ using the map $(x_1, \dots, x_{n+1}) \rightarrow [x_1, \dots, x_{n+1}, 1]$. Note that $S^n(k) = V_k \cap \mathbf{R}^{n+1} = \{(x_1, \dots, x_{n+1}) \in \mathbf{R}^{n+1} \mid x_1^{2k} + \dots + x_{n+1}^{2k} = 1\}$ is a nonsingular real algebraic set diffeomorphic to S^n . Moreover,

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the Zariski closure of $S^n(k)$ in $\mathbf{CP}^{n+1}$ coincides with V_k . It follows that if $S^n(k)$ and $S^n(l)$ are birationally equivalent, then so are V_k and V_l . Hence $\Omega_{\text{alg}}(S^n)$ is an infinite set.

The following conjecture seems to be reasonable.

CONJECTURE. For every compact smooth manifold M , the set $\Omega_{\text{alg}}(M)$ is infinite.

We shall prove this conjecture in some special cases.

THEOREM. *Let M be a smooth compact hypersurface in $\mathbf{R}^{n+1}$, $n \geq 1$. Then there exists a family $\{X_k\}$, $k = 1, 2, \dots$, of nonsingular algebraic subsets of $\mathbf{R}^{n+1}$ such that X_k is isotopic to M and for $k \neq l$, X_k and X_l are not birationally equivalent. In particular, the set $\Omega_{\text{alg}}(M)$ is infinite.*

PROOF. The proof is a refinement of the idea applied above for S^n . We identify $\mathbf{R}^{n+1}$ with a subset of the real projective space $\mathbf{RP}^{n+1}$ via the map $(x_1, \dots, x_{n+1}) \rightarrow [x_1, \dots, x_{n+1}, 1]$. Since M bounds in $\mathbf{RP}^{n+1}$, there exists a smooth function $f: \mathbf{RP}^{n+1} \rightarrow \mathbf{R}$ with 0 in $\mathbf{R}$ being its regular value and $M = f^{-1}(0)$. Let $p: S^{n+1} \rightarrow \mathbf{RP}^{n+1}$ be the standard projection. Fix a convex neighborhood $\mathcal{U}$ of $f \circ p$, in the space of all smooth functions from S^{n+1} to $\mathbf{R}$ equipped with the C^∞ topology, such that for each g in $\mathcal{U}$ the function $S^{n+1} \rightarrow \mathbf{R}$, defined by $x \rightarrow g(-x)$ for x in S^{n+1} , also belongs to $\mathcal{U}$ and the set

$$H(g) = \{[x] \in \mathbf{RP}^{n+1} \mid \tfrac{1}{2}(g(x/\|x\|) + g(-x/\|x\|)) = 0\}$$

is a smooth submanifold of $\mathbf{R}^{n+1}$ isotopic to M in $\mathbf{R}^{n+1}$. By the Weierstrass approximation theorem, there exists a polynomial map $\varphi: \mathbf{R}^{n+2} \rightarrow \mathbf{R}$ whose restriction to S^{n+1} belongs to $\mathcal{U}$. Let $\psi(x) = \frac{1}{2}(\varphi(x) + \varphi(-x))$ for x in $\mathbf{R}^{n+1}$ and let d be the degree of ψ . Obviously, d is even.

Now for each positive integer k , define a function $f_k: \mathbf{R}^{n+2} \rightarrow \mathbf{R}$ by

$$f_k(x) = \begin{cases} \|x\|^{d+2(k+d)}\psi(x/\|x\|) & \text{for } x \in \mathbf{R}^{n+2} \setminus \{0\}, \\ 0 & \text{for } x = 0. \end{cases}$$

Note that f_k is a homogeneous polynomial of even degree $d + 2(k + d)$. By a small perturbation of f_k in the space of all homogeneous polynomials of degree $d + 2(k + d)$, one can obtain a homogeneous polynomial $g_k: \mathbf{R}^{n+2} \rightarrow \mathbf{R}$ of degree $d + 2(k + d)$ such that the restriction of g_k to S^{n+1} belongs to $\mathcal{U}$ and 0 in the complex space $\mathbf{C}^{n+2}$ is the only critical point of the complexification $q_{k,\mathbf{C}}$ of g_k . This is possible since in the vector space of all homogeneous real polynomials in $n + 1$ variables of degree at least 2, the set of polynomials having 0 in $\mathbf{C}^{n+2}$ as the only critical point is dense (cf. for instance [3, proof of Proposition 1]). By construction, $g_k(x) = g_k(-x)$, the degree of g_k being even. It follows that

$$X_k = \{[x] \in \mathbf{RP}^{n+1} \mid g_k(x) = 0\} = H(g_k|S^{n+1})$$

is a nonsingular algebraic subset of $\mathbf{R}^{n+1}$ which is isotopic to M in $\mathbf{R}^{n+1}$. The Zariski (complex) closure $X_{k,\mathbf{C}}$ of X_k in the complex projective space $\mathbf{CP}^{n+1}$ is a nonsingular complex projective hypersurface whose homogeneous ideal is generated by $g_{k,\mathbf{C}}$. Therefore the arithmetic genus $p_a(X_{k,\mathbf{C}})$ of $X_{k,\mathbf{C}}$ is equal to $\binom{d+2(k+n)-1}{n+1}$ (cf. [4, p. 54]). If X_k and X_l are birationally equivalent, then so are $X_{k,\mathbf{C}}$ and $X_{l,\mathbf{C}}$ (as complex varieties) and hence $p_a(X_{k,\mathbf{C}}) = p_a(X_{l,\mathbf{C}})$ (cf. [4, p. 413, Remark 5.6.1]). Since $d + 2(k + n) \geq n + 2$, the last equality implies $k = l$.

COROLLARY. *Let M be one of the smooth manifolds listed below:*

(i) *M is the connected sum of a compact smooth hypersurface N in $\mathbf{R}^{n+1}$ and k copies of the projective space $\mathbf{R}P^n$, $n \geq 1$.*

(ii) *M is a compact surface.*

Then $\Omega_{\text{alg}}(M)$ is an infinite set.

PROOF. (i) First observe that if X_i is a nonsingular real algebraic set, Y_i an algebraic subset of X_i and $B(X_i, Y_i)$ the algebraic blowing up of X_i along Y_i , $i = 1, 2$, then X_1 is birationally equivalent to X_2 if and only if $B(X_1, Y_1)$ is birationally equivalent to $B(X_2, Y_2)$.

Now if X is a nonsingular real algebraic set diffeomorphic to N , then by blowing up k points in X , one obtains a nonsingular real algebraic set diffeomorphic to the connected sum of N and k copies of $\mathbf{R}P^n$. Hence (i) follows from the theorem.

(ii) is a particular case of the theorem if M is orientable and a particular case of (i) if M is nonorientable.

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