UNRAMIFIED WHITTAKER FUNCTIONS ON THE METAPLECTIC GROUP

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ABSTRACT. Kazhdan (unpublished), Shintani [Sh] and Casselman and Shalika [CS] computed explicitly the unramified Whittaker function of a quasi-split p -adic group. This is the main local ingredient used in the Rankin-Selberg-Shimura method, which yielded interesting results in the study of Euler products such as $L(s, \pi \otimes \pi')$ by Jacquet and Shalika [JS] (here π, π' are cuspidal $GL(n, A_F)$ -modules), and $L(s, \pi, r)$ by [F] (here π is a cuspidal $GL(n, A_E)$ -module, E is a quadratic extension of the global field F , and r is the twisted tensor representation of the dual group of $\text{Res}_{E/F} GL(n)$). Our purpose here is to generalize Shintani's computation [Sh] from the context of $GL(n)$ to that of the metaplectic r -fold covering group $\tilde{G}$ of $GL(n)$ (see [F', FK]).

Notations. Let F be a nonarchimedean local field with a ring R of integers and a uniformizer u of the maximal ideal of R . Denote by q the cardinality of the residue field $R/(u)$ of F . Let r, n be positive integers. Put $G = GL(n, F), K = GL(n, R)$. Let μ_r be the cyclic group of order r . Denote by $\tilde{G}$ the r -fold central topological covering group of G (see [FK]). Then there is an exact sequence $1 \rightarrow \mu_r \xrightarrow{i} \tilde{G} \xrightarrow{p} G \rightarrow 1$, with a preferred section $s: G \rightarrow \tilde{G}$ of p . We identify μ_r with its image via i . We also fix an embedding of μ_r in the field $\mathbb{C}$ of complex numbers. Suppose that r is a unit in R (its valuation is one). Then K embeds (see [FK]) as a subgroup of $\tilde{G}$; we identify K with its image. Fix a Haar measure on G by the requirement that the volume of K is one.

Let $L_c(\tilde{G}/K)$ denote the commutative convolution algebra (see [FK]) of complex-valued compactly-supported K -biinvariant anti-genuine functions on $\tilde{G}$. A function $f: \tilde{G} \rightarrow \mathbb{C}$ is called anti-genuine if $f(\zeta g) = \zeta^{-1} f(g)$ for all ζ in μ_r and g in $\tilde{G}$. In writing ζg we used the embedding of μ_r in $\tilde{G}$; in writing $\zeta^{-1} f(g)$ we used the embedding of μ_r in $\mathbb{C}^\times$. Let π be an irreducible representation of $\tilde{G}$ which is unramified (has a nonzero K -fixed vector) and genuine ($\pi(\zeta g) = \zeta \pi(g)$; ζ in μ_r, g in $\tilde{G}$). By the theory of the Satake transform (see [FK]) it determines an algebra homomorphism, denoted again by π , of $L_c(\tilde{G}/K)$ into $\mathbb{C}$.

For any n -tuple $m = (m_1, \dots, m_n)$ of integers, denote by u^m the diagonal matrix whose i th diagonal entry is u^{m_i} ($1 \leq i \leq n$). Denote by N the group of upper triangular unipotent matrices in G . The section s injects N and $u^{\mathbf{r}\mathbf{Z}}$ as subgroups of $\tilde{G}$ (see [FK]). We identify N and $u^{\mathbf{r}\mathbf{Z}}$ with their images in $\tilde{G}$. Write $m(i)$ for

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$m = (1, \dots, 1, 0, \dots, 0)$ where 1 appears in the first i places. Let f_i ($1 \leq i \leq n$) be the member of $L_c(\tilde{G}/K)$ which is supported on $\mu_r Ku^{rm(i)}K$ and attains the value 1 at $u^{rm(i)}$. Then $L_c(\tilde{G}/K)$ is isomorphic to the polynomial ring generated by $f_1, \dots, f_n$.

Choose n complex numbers $t_1, \dots, t_n$ so that the i th elementary symmetric function $\text{Sym}_i((t_j)) = \sum_{j_1 \leq \dots \leq j_i} t_{j_1} \cdots t_{j_i}$ in the t_j 's is equal to $q^{ir(i-1)/2} \pi(f_i)$ ($1 \leq i \leq n$). Let t be the diagonal matrix whose i th diagonal entry is t_i ($1 \leq i \leq n$). It lies in $GL(n, \mathbb{C})$ since $\det t = q^{nr(n-1)/2} \pi(f_n) \neq 0$.

Let $c(t)$ be the complex-valued function on $\mathbb{Z}^n$ which attains the value zero at $m = (m_1, \dots, m_n)$ unless $m_1 \geq \dots \geq m_n$, where

$$c(t; m) = \det(t_i^{m_j + n - j}; 1 \leq i, j \leq n) / \det(t_i^{n-j}; 1 \leq i, j \leq n).$$

Here the numerator is called a Schur function (see [M, p. 24]), and it is divisible in $\mathbb{Z}[t_1, \dots, t_n]$ by the denominator, which is the Vandermonde determinant $\prod (t_i - t_j)$ ($1 \leq i < j \leq n$). Note that for m with $m_1 \geq \dots \geq m_n$, $c(t; m)$ is the value at t of the character of the irreducible representation of $GL(n, \mathbb{C})$ with highest weight m . We have $c(t; m(i)) = \text{Sym}_i((t_j))$, and this is equal to $q^{ir(i-1)/2} \pi(f_i)$ by the definition of t .

Choose a character ψ of the additive group of F which is trivial on R but not on $u^{-1}R$. Denote by ψ also the character of N given by $\psi(x) = \prod_{i=1}^{n-1} \psi(x_{i,i+1})$, where $x_{i,i+1}$ is the $(i, i+1)$ entry of x . Given π and ψ , the function W on $\tilde{G}$ is called an *unramified Whittaker function* associated with π and ψ if it satisfies

$$(1) \quad W(\varsigma x g k) = \varsigma \psi(x) W(g) \quad (x \text{ in } N, \varsigma \text{ in } \mu_r, g \text{ in } \tilde{G}, k \text{ in } K),$$

and

$$(2) \quad \pi(f)W(g) = \int_G W(gh)f(h)dh \quad (f \text{ in } L_c(\tilde{G}/K), g \text{ in } \tilde{G}).$$

The integral is taken over $G \simeq \tilde{G}/\mu_r$; the integrand is invariant under μ_r . Let D be the set of m with $m_1 \geq \dots \geq m_n \geq 0$ and $r + m_i < m_{i+1}$ ($1 \leq i < n$). For each d in D put $a(d; m) = W(s(u^d)u^r m)$. Recall that we identify $u^{\mathbb{Z}}$ with its image in $\tilde{G}$ via the section s . Since $G = \bigcup Nu^m K$ (disjoint union over m in $\mathbb{Z}^n$), to determine W on $\tilde{G}$ it suffices (by (1)) to evaluate $a(d; m)$ for all d in D and m in $\mathbb{Z}^n$. Since the conductor of ψ is R , it follows from (1) that $a(d; m)$ is zero unless $m_1 \geq m_2 \geq \dots \geq m_n$.

THEOREM. *For each d in D we have $a(d; m) = a(d; 0)q^r \sum_{i=1}^n (i-n)m_i c(t; m)$ for all m in $\mathbb{Z}^n$.*

Let $I(i)$ be the set of all n -tuples $e = (e_1, \dots, e_n)$ with entries e_i equal to zero or one such that $e_1 + \dots + e_n = i$. Put $N_R = N \cap K$ and $N_R(e) = N_R \cap u^e K u^{-e}$. Note that the cardinality $[N_R/N_r(re)]$ of $N_R/N_r(re)$ is q^{rw} , where

$$w = \sum_{j>k} \max(e_j - e_k, 0) = in - i(i-1)/2 - \sum_{j=1}^n j e_j.$$

Denote by $f[xu^r e K]$ the right K -invariant anti-genuine complex-valued function on $\tilde{G}$ which is supported on $\mu_r x u^r e K$ (e in $I(i)$, x in $N_R/N_r(re)$), and attains the value one at $xu^r e$.

We first assume the validity of the following lemma.

LEMMA. For each $1 \leq i \leq n$ we have $f_i = \sum_{e \in I(i)} \sum_{x \in N_R/N_R(re)} f[xu^{re}K]$.

The Lemma implies that for $m = (m_1, \dots, m_n)$ in $\mathbf{Z}^n$ with $m_1 \geq \dots \geq m_n$ we have

$$\begin{aligned} \pi(f_i)a(d; m) &= \int_G W(s(u^d)u^{rm}g)f_i(g) dg \\ &= \sum_e [N_R/N_R(re)]a(d; m+e) \quad (e \text{ in } I(i)) \\ &= q^{irn-ir(i-1)/2} \sum_e q^{-r} \sum_{j=1}^n j^{e_j} a(d; m+e). \end{aligned}$$

Put

$$b(d; m) = q^r \sum_{j=1}^n (n-j)m_j a(d; m).$$

If $m_1 \geq \dots \geq m_n$, then we have

$$q^{ir(i-1)/2} \pi(f_i)b(d; m) = \sum_e b(d; m+e) \quad (1 \leq i \leq n).$$

Otherwise $b(d; m) = 0$. Namely $b(d; m)$ satisfies the equation

$$c(t; m(i))b(d; m) = \sum_e b(d; m+e) \quad \text{if } m_1 \geq \dots \geq m_n.$$

On the other hand, the function $c(t; j)$ satisfies the equation

$$c(t; m(i))c(t; m) = \sum_e c(t; m+e) \quad \text{if } m_1 \geq \dots \geq m_n.$$

Hence both $c(t; m)$ and $b(d; m)$ (for each d in D) are functions of m in $\mathbf{Z}^n$ which satisfy the same system of difference equations which has a unique solution up to a constant multiple. Since $c(t; 0) = 1$ the theorem follows.

It remains to prove the Lemma.

PROOF OF LEMMA. Let W be the Weyl group of permutation matrices in K , realized as a group of matrices with entries zero and one only. Let I be the Iwahori subgroup of K which consists of all matrices in K whose under diagonal entries are all in uR . Put $\bar{N}$ for the group of lower triangular unipotent matrices, N_I for $\bar{N} \cap I$, and A_R for the diagonal subgroup of K . We have the decompositions $I = N_R A_R \bar{N}_I = \bar{N}_I A_R N_R$ and $K = \bigcup N_R \bar{N}_I w N_R A_R$ (disjoint union over w in W). Put $e(i)$ for $m = (0, \dots, 0, 1, \dots, 1)$, where 1 appears in the last i entries. Since $u^{-re(i)} N_R u^{re(i)}$ lies in K , we have

$$K u^{rm(i)} K = K u^{re(i)} K \subset \bigcup_{w \in W} N_R A_R \bar{N}_I w u^{re(i)} w^{-1} K.$$

Put y for the element $w u^{re(i)} w^{-1}$ of $\tilde{G}$. For $1 \leq j < k \leq n$, and a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, write $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{jk}$ for the matrix (x_{uv}) in $GL(n)$ whose entries along the diagonal are one except that $x_{jj} = a$, $x_{kk} = d$, and its nondiagonal entries are zero except that $x_{jk} = b$ and $x_{kj} = c$.

Suppose that $\bar{n} = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}_{jk}$ is a matrix in $\bar{N}_I$, and $y^{-1} \bar{n} y$ does not lie in K . Write $|\cdot|$ for the valuation on F normalized by $|u| = q^{-1}$. Then $q^{-r} < |x| < 1$, the

j th diagonal entry of $p(y)$ is one and the k th is u^r , and

$$y^{-1}\bar{n}y = \begin{pmatrix} 1 & 0 \\ u^{-r}x & 1 \end{pmatrix}_{jk}.$$

We have

$$\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}_{jk} = \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}_{jk} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{jk} \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}_{jk} \begin{pmatrix} 1 & x^{-1} \\ 0 & 1 \end{pmatrix}_{jk}.$$

Given the matrix $s\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_{jk}\right)$ in $\tilde{G}$, we have in $\tilde{G}$ (see [FK, (2.1)]) the relation

$$\begin{aligned} s\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_{jk}\right) s\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}_{jk}\right) s\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}_{jk}\right)^{-1} \\ = s\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}_{jk}\right) \left(\frac{b}{a, x}\right). \end{aligned}$$

Here $(\cdot, \cdot)$ denotes the nondegenerate bimultiplicative r th Hilbert symbol

$$F^\times / F^{\times r} \times F^\times / F^{\times r} \rightarrow \mu_r.$$

Taking $b = 1$ and a unit a , namely $\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}_{jk}$ in K , we conclude that the anti-genuine K -biinvariant function f_i attains the value zero at $\bar{n}y$, since

$$\begin{aligned} f_i(\bar{n}y) &= f_i\left(y s\left(\begin{pmatrix} 1 & x^{-1}u^r \\ 0 & 1 \end{pmatrix}_{jk}\right) s\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_{jk}\right) \right. \\ &\quad \left. \times s\left(\begin{pmatrix} xu^{-r} & 0 \\ 0 & x^{-1}u^r \end{pmatrix}_{jk}\right) s\left(\begin{pmatrix} 1 & x^{-1}u^r \\ 0 & 1 \end{pmatrix}_{jk}\right) \right) \\ &= f_i\left(s\left(\begin{pmatrix} xu^{-r} & 0 \\ 0 & x^{-1}u^r \end{pmatrix}_{jk}\right) y\right) \\ &= f_i\left(s\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}_{jk}\right) s\left(\begin{pmatrix} xu^{-r} & 0 \\ 0 & x^{-1}u^r \end{pmatrix}_{jk}\right) y\right) \\ &= (a, x) f_i\left(s\left(\begin{pmatrix} xu^{-r} & 0 \\ 0 & x^{-1}u^r \end{pmatrix}_{jk}\right) y s\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}_{jk}\right)\right) \\ &= (a, x) f_i\left(s\left(\begin{pmatrix} xu^{-r} & 0 \\ 0 & x^{-1}u^r \end{pmatrix}_{jk}\right) y\right) \\ &= (a, x) f_i(\bar{n}y). \end{aligned}$$

Since the Hilbert symbol $(\cdot, \cdot)$ is nondegenerate we can find a unit a with $(a, x) \neq 1$; indeed, $(a, b) = 1$ for any pair a, b of units, hence there exists some unit a with $(a, u) = \zeta$, where ζ is a primitive r th root of unity. It follows that f_i is supported on the subset

$$\mu_r \bigcup_{w \in W} N_R w u^{re(i)} w^{-1} K \quad \text{of } \tilde{G}.$$

Since this set is contained in $\mu_r K u^{m(i)} K$, the Lemma follows.

REMARK. If π has a Whittaker model, namely we have $\pi(g)W(h) = W(hg)$ for all h, g in $\tilde{G}$, then $a(d; m) = \pi(s(u^d)u^{rm})a(0; 0)$, and in particular $a(d; 0) = \pi(s(u^d))a(0; 0)$ for all d in D . In this case there exists a unique (up to a scalar multiple) unramified Whittaker function associated with π, ψ .

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