A NECESSARY AND SUFFICIENT CONDITION FOR w^* -BOUNDED SETS TO BE STRONGLY BOUNDED

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ABSTRACT. It is known that if a locally convex space E is quasi-complete then every $\sigma(E', E)$ -bounded set in E' is $\beta(E', E)$ -bounded. This result cannot be reversed. Here we show that every $\sigma(E', E)$ -bounded set is $\beta(E', E)$ -bounded iff E is fast complete.

Throughout the paper E is a locally convex space. In accordance with [1] we call a balanced convex set $B \subset E$ a disk. For a disk $B \subset E$ we denote by E_B the linear hull of B and provide it with the topology generated by the basis $\{(1/n)B; n \in \mathbb{N}\}$. If the normed space E_B is Banach, we call B a Banach disk. If every set bounded in E is contained in some bounded Banach disk, the space E is called fast complete. It is easy to show that every quasi-complete locally convex space is fast complete.

It is well known [2, IV.5.3], that if a locally convex space E is quasi-complete then $\sigma(E', E)$ bounded subsets of E' are $\beta(E', E)$ -bounded. This result can be slightly generalized as follows. Assume E to be fast complete and take a bounded set $A \subset E$. Then A is contained in some bounded Banach disk B . For any $\sigma(E', E)$ -bounded set $H \subset E'$, the set $\{f(x); f \in H\}$ is bounded for any x from the Banach space E_B . Since A is bounded in E_B , the set $\{f(x); f \in H, x \in A\}$ is bounded by the Uniform Boundedness Principle and H is $\beta(E', E)$ -bounded.

THEOREM 1. *Let E be a locally convex space. Then all $\sigma(E', E)$ -bounded sets are $\beta(E', E)$ -bounded iff E is fast complete.*

PROOF. One implication has already been proved. Assume that E is not fast complete. Then there exists a bounded and closed disk $B \subset E$ such that E_B is not complete. Let $\{x_n\}$ be a Cauchy sequence in E_B which does not converge in E_B . Since the set $A_0 = \{x_n; n \in \mathbb{N}\}$ is bounded in E_B and B is a 0-neighborhood, we may assume $x_n \in B$, $n \in \mathbb{N}$.

From [3, Lemma 6-1-11], follows

CLAIM 1. $\{x_n\}$ does not converge in E .

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The Cauchy sequence $\{x_n\}$ is bounded and does not have any limit. Hence it cannot be contained in a finite-dimensional space. By taking a subsequence we may assume that the set A_0 is linearly independent. Take $H_0 \subset E$ such that $A_0 \cup H_0$ is a Hamel basis for E and denote by F the linear hull of H_0 .

There exists a balanced convex 0-neighborhood U in E such that $x_n \notin U$ for all $n \in N$. Choose and keep for the rest of the proof a balanced convex 0-neighborhood V in E such that $V + V \subset U$. Since $V \cap E_B$ is a 0-neighborhood in E_B , there exists $n_0 \in N$ such that $x_m - x_n \in V$ for $m, n > n_0$. If we discard the first n_0 terms of $\{x_n\}$ and renumber the sequence, we may assume $x_n \in x_1 + V$, $n \in N$.

CLAIM 2. Let A be the convex hull of A_0 . Then $A \cap V = \emptyset$.

Assume $x \in A \cap V$. Then there are $\lambda_k \geq 0$, $\sum \lambda_k = 1$ such that $x = \sum \lambda_k x_k = x_1 + \sum \lambda_k (x_k - x_1) \in x_1 + V$ and $x_1 \in x - v \subset V + V \subset U$, a contradiction.

CLAIM 3. Let W be a convex hull of $V \cup F$. Then $A \cap W = \emptyset$.

Assume $x \in A \cap W$. We may write $x = \lambda y + (1 - \lambda)z$, where $\lambda \in [0, 1]$, $y \in V$, $z \in F$. Further, $y = \sum \lambda_k x_k + \sum \mu_\alpha h_\alpha$ and $x - \lambda \sum \lambda_k x_k = \lambda \sum \mu_\alpha h_\alpha + (1 - \lambda)z \in E_A \cap F_1$, a contradiction.

The set W is a balanced convex 0-neighborhood in E . Let p be its Minkowski functional. For any $n \in N$ denote by f_n the linear function on E such that $f_n(x) = 0$ for $x \in H_0 \cup (A_0 \setminus x_n)$ and $f_n(x_n) = 1$. By Claim 2, for any $x \in A$ we have $p(x) \geq 1$ and $|f_n(x)| = |f_n(\sum \lambda_k x_k)| = |\lambda_n| \leq 1 \leq p(x)$. Hence also $|f_n(x)| \leq p(x)$ for $x \in E_A$. For $x \in F$, we have $p(x) = 0$ and $f_n(x) = 0$. Finally, if $x \in E$, then $x = y + z$, where $y \in E_A$, $z \in F$, and

$$p(x) = p(y + z) \leq p(y) + p(z) = p(y) = p(x - z) \leq p(x) + p(z) = p(x).$$

Therefore $|f_n(x)| = |f_n(y)| \leq p(y) = p(x)$ and f_n is a continuous linear functional on E .

Put $S = \{nf_n; n \in N\}$ and choose $x \in E$. We can write $x = y + \sum_{k=1}^m \xi_k x_k$, where $y \in F$. Since for any $n > m$, $f_n(x) = 0$ and $\sup\{|nf_n(x)|; n \in N\} = \sup\{n\xi_n; n = 1, 2, \dots, m\} < +\infty$, the set S is $\sigma(E', E)$ -bounded.

On the other hand, the set A_0 , bounded in E_B , is also bounded in E and we have

$$\sup\{|nf_n(x)|; n \in N, x \in A_0\} \geq \sup\{|nf_n(x_n)|; n \in N\} = +\infty,$$

i.e. S is not $\beta(E', E)$ -bounded.

THEOREM 2. *Let E be a locally convex space. Then all $\sigma(E, E')$ -bounded sets are $\beta(E, E')$ -bounded iff E is fast complete.*

PROOF. The $\sigma(E, E')$ -bounded sets are $\beta(E, E')$ -bounded iff the $\sigma(E', E)$ -bounded sets are $\beta(E', E)$ -bounded [3, 10.5.4]. Hence Theorem 2 follows from Theorem 1.

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